On Unitary Transform Approximations

Ricardo L. de Queiroz, *Member, IEEE*

***Abstract*—In this letter, we present a method to find a unitary transform which is the closest to a given square transform matrix in the sense of minimizing the variance of the output error (discrepancy) for given signal statistics. An extension of the result to frequency responses of filterbank transfer matrices is also given along with an example to demonstrate the feasibility of the method.**

***Index Terms*—Filterbanks, transform.**

I. Motivation

UNITARY matrices have several properties, such as norm preservation, that make them attractive for numerous applications, e.g., image compression, adaptive filtering, etc. [1]. In some circumstances, it is desirable to find a unitary matrix that best approximates a given square matrix. Those transforms are useful in image compression and to construct time-varying filterbanks [2], [3]. In [4], it is shown how to minimize the norm of the difference between said matrices. In this letter, we are interested in transforms which are used to transform blocks of samples from a wide-sense stationary signal with known statistics. Thus, we are interested in minimizing the output error incurred by replacing one matrix by another. In this letter, we present a solution for the problem and also include an extension to approximate a paraunitary filterbank [5] (a transform whose polyphase transfer matrix is unitary for $z = e^{j\omega}$). A simple design example is also presented.

II. Unitary Approximation

Let the zero-mean input signal $x(n)$ be transformed by the $M \times M$ matrix $\boldsymbol{A}$ on a block-by-block basis as $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{x}$ represents a block of M input samples. Assume the signal $x(n)$ to have autocorrelation matrix $\boldsymbol{R}_{xx}$. Let $\boldsymbol{B}$ be a unitary matrix of the same size and define an error vector as

$$\epsilon = \boldsymbol{y} - \boldsymbol{B}\boldsymbol{x} = (\boldsymbol{A} - \boldsymbol{B})\boldsymbol{x}. \tag{1}$$

Theorem 1: The unitary matrix $\boldsymbol{B}$ which is the closest to the nonsingular matrix $\boldsymbol{A}$ in the sense of minimizing the distance $J = E[\epsilon^H \epsilon]$ (error energy) is given by $\boldsymbol{B} = \boldsymbol{Q}_1\boldsymbol{Q}_2$, where $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are the unitary matrices derived from the singular value decomposition (SVD) of $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{R}_{xx}$ as $\boldsymbol{C} = \boldsymbol{Q}_1\boldsymbol{\Lambda}\boldsymbol{Q}_2$.

Proof: $J = E[\epsilon^H \epsilon] = E[\boldsymbol{x}^H(\boldsymbol{A}-\boldsymbol{B})^H(\boldsymbol{A}-\boldsymbol{B})\boldsymbol{x}]$. By exchanging the inner product by the trace of the outer product we have

$$\begin{aligned} J &= E[\mathrm{Tr}\{(\boldsymbol{A}-\boldsymbol{B})\boldsymbol{x}\boldsymbol{x}^H(\boldsymbol{A}-\boldsymbol{B})^H\}] \\ &= \mathrm{Tr}\{(\boldsymbol{A}-\boldsymbol{B})\boldsymbol{R}_{xx}(\boldsymbol{A}-\boldsymbol{B})^H\} \\ &= \mathrm{Tr}\{\boldsymbol{A}\boldsymbol{R}_{xx}\boldsymbol{A}^H + \boldsymbol{B}\boldsymbol{R}_{xx}\boldsymbol{B}^H - \boldsymbol{B}\boldsymbol{R}_{xx}\boldsymbol{A}^H \\ &\quad - \boldsymbol{A}\boldsymbol{R}_{xx}\boldsymbol{B}^H\}. \end{aligned} \tag{2}$$

In (2), note that 1) $\mathrm{Tr}\{\boldsymbol{B}\boldsymbol{R}_{xx}\boldsymbol{B}^H\} = E[\mathrm{Tr}\{\boldsymbol{B}\boldsymbol{x}\boldsymbol{x}^H\boldsymbol{B}^H\}] = E[\boldsymbol{x}^H\boldsymbol{B}^H\boldsymbol{B}\boldsymbol{x}] = E[\boldsymbol{x}^H\boldsymbol{x}]$, which is independent of $\boldsymbol{B}$; 2) $\boldsymbol{A}\boldsymbol{R}_{xx}\boldsymbol{A}^H$ is independent of $\boldsymbol{B}$; 3) since $\boldsymbol{R}_{xx}^H = \boldsymbol{R}_{xx}$, then $\boldsymbol{A}\boldsymbol{R}_{xx}\boldsymbol{B}^H = (\boldsymbol{B}\boldsymbol{R}_{xx}\boldsymbol{A}^H)^H$; and 4) for a square matrix $\boldsymbol{D}, \mathrm{Tr}\{\boldsymbol{D} + \boldsymbol{D}^H\} = 2\mathrm{Re}\,\mathrm{Tr}\{\boldsymbol{D}\}\}$. In light of these facts, we can see that the minimization of J is equivalent to the maximization of

$$J' = \mathrm{Re}\{\mathrm{Tr}\{\boldsymbol{C}\boldsymbol{B}^H\}\} \tag{3}$$

where $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{R}_{xx}$. Let the SVD of $\boldsymbol{C}$ be given by $\boldsymbol{C} = \boldsymbol{Q}_1\boldsymbol{\Lambda}\boldsymbol{Q}_2$, where the $\boldsymbol{Q}_i$ matrices are unitary and $\boldsymbol{\Lambda}$ is a diagonal matrix containing the nonnegative singular values of $\boldsymbol{C}$. Then

$$\begin{aligned} \mathrm{Tr}\{\boldsymbol{C}\boldsymbol{B}^H\} &= \mathrm{Tr}\{\boldsymbol{Q}_1\boldsymbol{\Lambda}\boldsymbol{Q}_2\boldsymbol{B}^H\} = \mathrm{Tr}\{\boldsymbol{\Lambda}\boldsymbol{Q}_2\boldsymbol{B}^H\boldsymbol{Q}_1\} \\ &= \mathrm{Tr}\{\boldsymbol{\Lambda}\boldsymbol{Q}\} \end{aligned}$$

where $\boldsymbol{Q} = \boldsymbol{Q}_2\boldsymbol{B}^H\boldsymbol{Q}_1$ is some unitary matrix. Let the entries of $\boldsymbol{Q}$ be q_{ij}, and the singular values of $\boldsymbol{C}$ (diagonal entries in $\boldsymbol{\Lambda}$) be denoted by v_i. Hence

$$J' = \mathrm{Re}\left[\sum_{i=0}^{M-1} v_i q_{ii}\right]. \tag{4}$$

Furthermore, if $\boldsymbol{A}$ is nonsingular, so is $\boldsymbol{C}$ and all v_i are positive and real. Therefore, since $\boldsymbol{Q}$ is unitary and $\Sigma_j |q_{ij}|^2 = 1$, (4) is maximized iff $q_{ii} = 1$ and $q_{ij} = 0$ for $i \neq j$, i.e., $\boldsymbol{Q} = \boldsymbol{I}_M$. As a result, $\boldsymbol{Q}_2\boldsymbol{B}^H\boldsymbol{Q}_1 = \boldsymbol{I}_M$ and $\boldsymbol{B}^H = \boldsymbol{Q}_2^H\boldsymbol{Q}_1^H$. Thus

$$\boldsymbol{B} = \boldsymbol{Q}_1\boldsymbol{Q}_2. \quad \square \tag{5}$$

Note that, in general, this result is quite different from the matrix approximation problem [4], which becomes a special case. For an uncorrelated (white) signal, then $\boldsymbol{R}_{xx} = \boldsymbol{I}_M$ and the the above result matches the one in [4]. Under another view, we can say that, for uncorrelated input, the error variance cost can be mapped to a matrix norm formulation.

Corollary 1: If $\boldsymbol{A}$ is a singular matrix, a unitary matrix $\boldsymbol{B}$ which minimizes $E[\epsilon^T \epsilon]$, given in (5), is a nonunique solution. If all v_i are sorted by magnitude, the general solution is given by $\boldsymbol{B} = \boldsymbol{Q}_1 \begin{bmatrix} \boldsymbol{I}_K & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix} \boldsymbol{Q}_2$, where $\mathrm{rank}(\boldsymbol{A}) = K$ and $\boldsymbol{E}$ is any $(M-K) \times (M-K)$ unitary matrix.

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The author is with Xerox Corporation, Webster, NY 14580 USA (e-mail: queiroz@wrc.xerox.com).

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Proof: If $\boldsymbol{A}$ is singular, $\boldsymbol{C}$ is singular and $v_i = 0, i \geq K$. Then, (4) becomes $\mathrm{Re}\{\Sigma_{i<K}\ v_i q_{ii}\}$, such that conditions $q_{ii} = 1$ and $q_{ij} = 0$ (for $i \neq j | \{j < K, \forall i\} \vee \{i < K, \forall j\}$) are necessary. However, for $i \geq K$, entries q_{ij} are irrelevant to (4). Thus, entries q_{ij} for $i \geq K$ and $j \geq K$ have to form a unitary matrix, from which $\boldsymbol{B} = \boldsymbol{Q}_1 \boldsymbol{Q}^H \boldsymbol{Q}_2 = \boldsymbol{Q}_1 \begin{bmatrix} \boldsymbol{I}_K & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix} \boldsymbol{Q}_2$ derives. As a result, $\boldsymbol{B} = \boldsymbol{Q}_1 \boldsymbol{Q}_2$ is a nonunique closest approximation. □

Corollary 2: For the case of a linear block transform $\boldsymbol{A}$ with real entries and $x(n)$ is a real signal, $\boldsymbol{B} = \boldsymbol{Q}_1 \boldsymbol{Q}_2$ is a real orthogonal matrix that approximates $\boldsymbol{A}$ in the sense of minimizing $E[\epsilon^T \epsilon]$.

Proof: In this case $\boldsymbol{C}$ is real and so are $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$, thus $\boldsymbol{B}$ is orthogonal. □

A. Extension to Paraunitary Filterbank Approximation

Let $\boldsymbol{A}(z)$ be the polyphase transfer matrix (PTM) of an M-channel biorthogonal filterbank. (See [5] for details of transfer matrices and filter banks.) The PTM relates the input vector $\boldsymbol{x}(z)$, containing the z-transforms $X_i(z)$ of the M polyphase components of $x(n)$, to the output vector $\boldsymbol{y}(z)$ which contains the z-transforms $Y_i(z)$ of the M subband signals, as $\boldsymbol{y}(z) = \boldsymbol{A}(z)\boldsymbol{x}(z)$. Assume $x(n)$ is periodic with a very large period N_p and let all Fourier transforms be computed over one period. It can be shown that

$$E[X_i(e^{j\omega}) X_j^H(e^{j\omega})] = N_p \Gamma_{ij}(e^{j\omega}) \tag{6}$$

where $\Gamma_{ij}(e^{j\omega})$ is the Fourier transform of the cross correlation between polyphases i and j, i.e., $\Gamma_{ij}(e^{j\omega}) = \mathcal{F}\{r_{xx}(m + i - j)\}$, where $r_{xx}(m)$ is the autocorrelation function of $x(n)$.

Corollary 3: Let $\boldsymbol{\Gamma}(e^{j\omega})$ have entries $\Gamma_{ij}(e^{j\omega})$. The paraunitary PTM $\boldsymbol{B}(z)$ which is the closest to $\boldsymbol{A}(z)$ in the sense of minimizing the distance $J = E[\epsilon^H(e^{j\omega})\epsilon(e^{j\omega})]$, where $\epsilon(e^{j\omega}) = \boldsymbol{y}(e^{j\omega}) - \boldsymbol{B}(e^{j\omega})\boldsymbol{x}(e^{j\omega})$, has its frequency response governed by

$$\boldsymbol{B}(e^{j\omega}) = \boldsymbol{Q}_1(e^{j\omega}) \boldsymbol{Q}_2(e^{j\omega}) \tag{7}$$

where $\boldsymbol{Q}_1(e^{j\omega})$ and $\boldsymbol{Q}_2(e^{j\omega})$ are unitary matrices derived from the SVD of

$$\boldsymbol{C}(e^{j\omega}) = \boldsymbol{A}(e^{j\omega})\boldsymbol{\Gamma}(e^{j\omega}) = \boldsymbol{Q}_1(e^{j\omega})\boldsymbol{\Lambda}(e^{j\omega})\boldsymbol{Q}_2(e^{j\omega}). \tag{8}$$

Proof: From (6), we have $E[\boldsymbol{x}(e^{j\omega})\boldsymbol{x}^H(e^{j\omega})] = N_p \boldsymbol{\Gamma}(e^{j\omega})$. Since N_p does not affect the minimization, the result in (7) follows from the proof of Theorem 1, where matrix entries are now a function of $e^{j\omega}$, and $\boldsymbol{R}_{xx}$ is replaced by $\boldsymbol{\Gamma}(e^{j\omega})$. □

III. EXAMPLE

In a simple example, suppose one wants to design a 8 × 8 unitary transform which has the lowpass bases with as much energy compaction as the discrete cosine transform (DCT) [1], but with higher frequency bases with as much space localization as the Haar transform. First, we construct an 8 × 8 nonunitary transform where four bases belong to the DCT and four to the Haar transform. Such matrix has a condition number close to two and the matrix entries are plotted in Fig. 1. Our unitary approximation for an AR(1) signal with $\rho = 0.95$ is also shown in Fig. 1. Applications will be presented in a forthcoming paper.

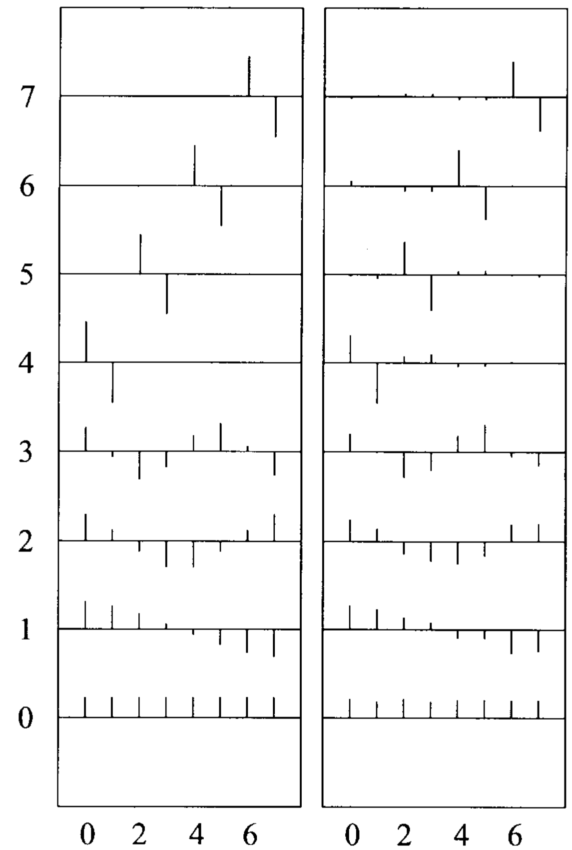

Fig. 1. Bases of a nonunitary matrix comprising DCT and Haar bases (left) and its closest unitary counterpart (right) for an AR(1) signal.

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