On Reconstruction Methods for Processing
Finite-Length Signals with Paraunitary Filter Banks

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Abstract—New expressions are developed for the perfect reconstruction of
the boundary regions of a finite-length signal after subband processing.
The time-invariant filter bank is required to be uniform and paraunitary,
using FIR filters regardless of phase or symmetry. They accommodate a
linear boundary extension in the analysis section, and avoid periodic
extensions or storage of extended subband signals. The reconstruction
methods are based on the formulation of linear systems that are built as
a function of the filters.

I. INTRODUCTION

The theory of multirate filter banks [1]-[3] often assumes infinite
length signals, while the problem of processing a finite length signal
has recently attracted the attention of some researchers [4]-[13].
Finite-length 1-D signals are frequently used to model images
separating transforms. Recently, they were also studied to construct
time-varying wavelet packets [14], [15]. The study of the boundary
distortion is nearly as old as the idea of subband coding of images,
including periodic extensions and the use of convolution in DFT
domain [4], inclusion of few extra samples [5], or perhaps a simple
study of which extension would minimize the border distortions [6].
However, it is quite easy to see that if the extension and the subband
filters are symmetric, the deleted subband samples could be recovered
by a simple symmetric extension of the subband signal [7]-[10].
In general, two-channel filter banks are assumed. In [10] and [11],
these results were extended for more than two channels, and in [11] a
reconstruction method was developed for nonlinear phase filters.
In [12], an alternative approach to [11] was proposed. Size-limited filter
banks are discussed in [13] and there are several proposals based on
applying special filter banks (basis functions) to the borders in order
to assure full orthogonality [14], [15].

Assume the finite-length signal \( x(n) \) has \( N_S \) samples and let
\( x = [x(0), \ldots, x(N_S - 1)]^T \). A nonexpansive analysis system will
convert \( x \) into \( N_S \) subband samples, which we similarly merge
into vector \( y \), for simplicity. After processing or quantization, the
resulting subband vector \( y \) is submitted to a synthesis system that
will recover the vector \( x \). Following [13], models for the size-limited
analysis or synthesis systems are shown in Fig. 1(a) and Fig. 1(b),
respectively. In these models, finite-length processing is accomplished
by converting the signal to a symmetric-periodic sequence, which is
processed and windowed. On the other hand, for real filter banks
there is a linear transform \( G \) such that \( y = Gx \). Thus, the perfect
reconstruction synthesis is accomplished by \( x = G^{-1}y \). However, it
is not always practical to invert a \( N_S \times N_S \) matrix, nor to perform
analysis or synthesis through a \( N_S \times N_S \) linear transform. We will
deal here with uniform paraunitary FIR filter banks, with arbitrary
phase response. The popular linear-phase filters are a particular
example, and although the results here can be surely applied in such
a case, they may offer a simpler solution because of the assumed
generality of the impulse responses of the filters. In addition, we
do not impose any symmetry restriction for the boundary extension
process. Under these conditions, given the boundary extension used
in the analysis section and the filter bank, we prove in the appendix
that assuming a linear boundary extension, \( G \) is always one-to-one
and onto, therefore, its inverse is unique. Also, all reconstruction
methods lead to identical results and have the same sensitivity to
quantization or processing of the subbands. We will limit ourselves
to the case where only time or subband samples are used to reconstruct
the original signal (while a combination of both can be found in [11]
and [12]) and assume \( y = x \).

In terms of notation, our conventions are: unidimensional concate-
nation of matrices and vectors is indicated by a comma; \([.]^T\) means
transposition, while \([.]^*\) and \([.]^H\) stand for pseudo-inverses of a
matrix, as \( A^* = (A^H A)^{-1} A^H \) and \( A^H = A^T (A A^T)^{-1} \). is the
\( n \times n \) identity matrix; \( 0_n \) is the \( n \times n \) null matrix; and \( J_n \) is the
\( n \times n \) counter-identity, reversing, or exchange matrix. For example

\[
J_4 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

II. TRANSFORM MATRIX OF A PARAUNITARY FILTER BANK

We assume an FIR uniform paraunitary filter bank (PUFB). We have
\( M \) analysis and synthesis filters. Let \( L \) be the maximum
number of taps and we define \( L \) to be a multiple of \( M \), as \( L = N_M \),
where \( N \) is the smaller integer such that the maximum filter length
lies between \( N_M \) and \( (N - 1)M \). If a filter’s impulse response
does not have length \( L \), we pad zeros so as to reach length \( L \).
The analysis and synthesis filters are, thus, denoted as \( f_k(n) \) and
\( g_k(n) \), respectively, \((k = 0, 1, \ldots, M - 1; n = 0, 1, \ldots, L - 1) \).
Let the input signal \( x(n) \) have its polyphase components denoted
as \( x_m(n) = x(nM + i) \) and the subband signals be denoted as
\( y_i(n) \). \((i = 0, 1, \ldots, M - 1; m \) integer\). Define a signal \( y(n) \)
whose polyphase components are the subband signals, i.e., \( y_i(n) =
y_i(nM + i) \). It is well known that a PUFB is a special form of a block
filter [1], governed by a paraunitary FIR transfer matrix \( E(z) \), which
relates the polyphase components of \( x(n) \) and \( y(n) \). Also, \( E(z) \) has
\( N \) impulse response matrices [1]. For a PUFB [2], [3] a transform
matrix \( P \) of size \( M \times L \) and elements \( p_{ij} \) \((i = 0, 1, \ldots, M - 1;
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ j = 0, 1, \ldots, L - 1 \) can be defined as

\[
p_{ij} = f_j(L - 1 - j) = g_i(j).
\]
P can be divided into $N$ square sub-matrices, yielding

$$\mathbf{P} = [\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{N-1}].$$

Therefore, we can express $\mathbf{P}$ as the impulse response matrices of $\mathbf{E}(z)$ as

$$\mathbf{E}(z) = \sum_{i=0}^{N-1} (\mathbf{P}_{N-1-i} \mathbf{M}_i) z^{-i}.$$  \hspace{1cm} (3)

The paraunitariness of $\mathbf{E}(z)$, i.e., $\mathbf{E}(z) \mathbf{E}^T(z^{-1}) = \mathbf{E}^T(z^{-1}) \mathbf{E}(z) = \mathbf{I}_M$, is also given by [2]

$$\sum_{i=0}^{N-1} \mathbf{P}_m \mathbf{P}_m^T = \sum_{m=0}^{N-1} \mathbf{P}_m \mathbf{P}_m^T = \delta(l) \mathbf{I}_M \hspace{1cm} (4)$$

where $\delta(l)$ is the Kronecker delta. We will use the matrix notation for the analysis and synthesis using PUFB's that basically follow a noncausal notation (see [1] for details on paraunitary systems and filter banks and [2] and [3] for details in using transform matrices in the description of PUFB's). Let $\mathbf{x}$ and $\mathbf{y}$ represent the time-domain and subband vectors of $N_S$ samples (containing signals $x(n)$ and $y(n)$). Using linear boundary extensions, $\mathbf{x}$ is first extended by $\lambda = (L - M)/2$ samples in each border, originating the vector $\hat{\mathbf{x}}$, which is, thus, transformed by matrix $\mathbf{T}$, resulting in the subband vector $\mathbf{y}$. Assume the signal has $N_S = N_B M$ samples, where $N_B$ is an integer, then $\mathbf{T}$ is a block-Toeplitz-like matrix [2], [3] given by

$$\mathbf{T} = \begin{bmatrix} \cdots & \mathbf{P}_0 & \mathbf{P}_1 & \cdots & \mathbf{P}_{N-1} \\ \mathbf{P}_0 & \mathbf{P}_1 & \cdots & \mathbf{P}_{N-1} \\ \vdots & \mathbf{P}_0 & \mathbf{P}_1 & \cdots & \mathbf{P}_{N-1} \end{bmatrix}.$$

Note that $\mathbf{T}$ has $N_B$ block-rows, is a nonsquare matrix ($N_S \times N_S + 2 \lambda$), and it cannot be inverted. Denoting by $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ the vectors in the synthesis section, analysis and synthesis are described by [2], [3]

$$\mathbf{y} = \mathbf{T} \mathbf{x} \hspace{1cm} \hat{\mathbf{x}} = \mathbf{T}^T \hat{\mathbf{y}}$$

The PR conditions in (4) tell us that $\mathbf{T}^T \mathbf{T} = \mathbf{I}_{N_S}$, but, obviously, $\mathbf{T}^T \mathbf{T} \neq \mathbf{I}$. If $\hat{\mathbf{y}} = \mathbf{y}$ (no processing or quantization), we have

$$\hat{\mathbf{x}} = \mathbf{T}^T \mathbf{T} \mathbf{x}.$$  \hspace{1cm} (7)

III. RECONSTRUCTING DISTORTED DATA

The analysis operation would require the knowledge of $\lambda$ samples outside the range of $\mathbf{x}$. Setting these samples to zero or using circular convolution can generate undesirable high-frequency components due to discontinuities at the borders. The extension of $\mathbf{x}$ into $\hat{\mathbf{x}}$ will be restricted here to be a linear boundary extension where the unknown samples are found by a linear combination of samples contained in $\mathbf{x}$. We assume that, for each border, the $\lambda$ samples across the border are found as a linear function of at most $\lambda$ boundary samples of the signal. We adopt the notation shown Fig. 2' where $\mathbf{x}'$ is divided into three regions as $\mathbf{x}' = [\mathbf{x}'_L, \mathbf{x}'_C, \mathbf{x}'_R]$, and the extended vector is formed by $\mathbf{x}' = [\mathbf{x}'_L, \mathbf{x}_L, \mathbf{x}'_C, \mathbf{x}_C, \mathbf{x}'_R, \mathbf{x}_R]$. Then, we have $\mathbf{x}' = [\mathbf{x}_L, \mathbf{x}_C, \mathbf{x}_R]$, where

$$\mathbf{x}_L = \mathbf{R}_L \mathbf{x}_c \quad \mathbf{x}_R = \mathbf{R}_r \mathbf{x}_r.$$  \hspace{1cm} (8)

The size of each subvector is indicated in Fig. 2 and $\mathbf{R}_L$ and $\mathbf{R}_r$ are arbitrary $\lambda \times \lambda$ matrices to extend the signal on the left and right borders, respectively. For example, a popular extension method is the symmetric extension [7], [10], which is mainly inherited from the use of linear-phase filters, i.e., $\mathbf{R}_L = \mathbf{R}_r = \mathbf{J}_\lambda$.

We will show solutions for the left border and the reader can easily infer the solution for the right border by simply reversing $\mathbf{x}$ and the columns of $\mathbf{P}$. In (7), using (4), we see that

$$\mathbf{H} = \mathbf{T}^T \mathbf{T} = \begin{bmatrix} \mathbf{H}_L & 0 \\ 0 & \mathbf{H}_R \end{bmatrix}.$$ \hspace{1cm} (9)

The matrix $\mathbf{H}$ is block diagonal and $\mathbf{H}_L$ and $\mathbf{H}_R$ are $2 \lambda \times 2 \lambda$ matrices. For the left border, if we let

$$\Phi = \begin{bmatrix} \mathbf{P}_0 & \mathbf{P}_1 & \cdots & \mathbf{P}_{N-2} \\ \mathbf{P}_0 & \mathbf{P}_1 & \cdots & \mathbf{P}_{N-3} \\ \vdots & \vdots & \mathbf{P}_0 \end{bmatrix}$$

then

$$\mathbf{H}_L = \Phi \mathbf{I} \Phi^T.$$ \hspace{1cm} (11)

From (7) and (9), and dividing $\mathbf{H}_L$ into two equal parts as $\mathbf{H}_L = [\mathbf{H}_L, \mathbf{H}_L]$, we have

$$\begin{bmatrix} \mathbf{x}_L \\ \mathbf{x}_C \\ \mathbf{x}_R \end{bmatrix} = [\mathbf{H}_L, \mathbf{H}_L] \begin{bmatrix} \mathbf{x}_L \\ \mathbf{x}_C \\ \mathbf{x}_R \end{bmatrix}.$$ \hspace{1cm} (12)

Hence, from (8)

$$\begin{bmatrix} \mathbf{x}_L \\ \mathbf{x}_C \\ \mathbf{x}_R \end{bmatrix} = ([\mathbf{H}_L, \mathbf{R}_L + \mathbf{H}_L] \mathbf{x}_L$$

and

$$\begin{bmatrix} \mathbf{x}_L \\ \mathbf{x}_C \\ \mathbf{x}_R \end{bmatrix} = ([\mathbf{H}_L, \mathbf{R}_L + \mathbf{H}_L] \mathbf{x}_L.$$ \hspace{1cm} (14)

Thus, $\mathbf{x}_L$ is recovered from the distorted extended signal using only linear relations.

IV. SUBBANDS EXTENSION TO PREVENT DISTORTION

The reconstruction problem is caused by the deletion of extra subband samples (resulting from the convolution with the subband filters) by the windowing process. There are $\lambda$ blocks of $M$ of these samples deleted from each border, where $K$ is the largest integer smaller than or equal to $N/2$. If we could infer these samples from the
samples actually retained along with the subbands, we could extend the subbands, use one algorithm for the whole synthesis section, and window the output without any distortion. As in the previous case, we present the solution for the left border, and the reader can infer the solution for the right border, by reversing x, the subband signals, and the columns of P. Let \( x_\text{in} = [x(0), x(1), \ldots, x(L-M-1)] \) and \( y_\text{in} = [y(0), y(1), \ldots, y(KM-1)] \). Note that \( y_\text{in} = [y(0), \ldots, y_{M-1}(0), y(1), \ldots, y_{M-1}(1), \ldots, y(M-1), \ldots, y(KM-1)] \). Let \( E_F \) and \( E_T \) be square matrices and \( x_\text{out} \) and \( y_\text{out} \) be vectors, such that

\[
y_{\text{out}} = E_T y_{\text{in}}, \quad \text{if} \quad x_{\text{out}} = E_F x_{\text{in}}. \quad (15)
\]

The physical meaning of these equations is: if \( x_{\text{out}} \) is an extension vector in time domain, then the extended subband signals are contained in \( y_{\text{out}} \). The relation between extended samples and samples actually existing in the signal is given by the matrices for extension in frequency (subbands) and time domains, \( E_F \) and \( E_T \), respectively. Let, also, \( F \) be a matrix similar to \( T \) but with only 2K block-rows, and assume \( F \) is divided into four \( K \times (N+2K-1)/2 \) submatrices.

\[
F = \begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix}.
\]

(16)

For example, for \( N = 3 \), we have \( F = \begin{pmatrix} P_0 & P_1 & P_2 & 0_M \\ 0_M & P_0 & P_1 & P_2 \end{pmatrix} \). We want to find a matrix \( E_F \) such that, for a given \( E_T \),

\[
\begin{bmatrix} y_{\text{out}} \\ y_{\text{in}} \end{bmatrix} = F \begin{bmatrix} x_{\text{out}} \\ x_{\text{in}} \end{bmatrix}.
\]

(17)

Thus

\[
\begin{bmatrix} E_F & y_{\text{in}} \\ y_{\text{in}} \end{bmatrix} = \begin{bmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{bmatrix} \begin{bmatrix} E_T & x_{\text{in}} \\ x_{\text{in}} \end{bmatrix} = \begin{bmatrix} F_{00} E_T + F_{01} \\ F_{10} E_T + F_{11} \end{bmatrix} x_{\text{in}}.
\]

(18)

The above equation carries two equalities. Substituting one in the other, we get

\[
E_T (F_{10} E_T + F_{11} ) x_{\text{in}} = (F_{00} E_T + F_{01}) x_{\text{in}}. \quad (19)
\]

Disregarding \( x_{\text{in}} \) and equating the matrices, we have one solution as

\[
E_T = (F_{00} E_T + F_{01}) (F_{10} E_T + F_{11})^{-1}.
\]

(20)

For the special case of linear-phase filters, \( E_T \) is simplified to a diagonal matrix with \( \pm 1 \) as the diagonal elements, according to the symmetry of the filters [11], having, thus, a trivial implementation. In this case, symmetric extensions are often applied [7]-[10].

V. CONCLUSION

We intended to show techniques to effectively achieve perfect reconstruction of the signal, regardless of the phase of the filters. Also, only a few restrictions are made on the boundary linear extensions. However, properties of these extensions are not explored here. We are also limited to uniform FIR PUTFs, but the same solution can be applied to nonuniform filter banks, given that they could be constructed by cascading uniform ones. It should be noted that all the solutions using linear extension and time-invariant\(^2\) filter banks have very different approaches but are identical in essence and results. Of course, this assumes that the entire computation uses real arithmetic or suffers negligible rounding effects, compared to the effects of subband quantization. So, sensitivity to quantization errors is only a function of the PUTF and of the extension method.

In order to compare our results with other results applicable to nonlinear-phase PUTF’s, we may rule out the direct inversion of \( G \) because of the amount of computation (for inversion and for the analysis or synthesis transformation) involved for large \( N_s \), even considering \( G \) is a relatively sparse matrix. Both solutions in [11] and [12], as well as those presented here, allow the use of any algorithm designed for the PUTF in question. However, the solutions in [11] and [12] lead to the evaluation of systems, for each border, of the form \( v = As + Bt \), where \( A \) and \( B \) are real matrices and \( v \) (\( \lambda \) elements), \( s \) (2\( K \)\( M \) elements), and \( t \) (2\( \lambda \) elements) are vectors corresponding to reconstructed samples, subband samples, and time-domain samples, respectively. [11] assumed \( N \) even, while [12] assumed \( N \) odd. Here, subband and time-domain samples are not computed together, which makes the implementation easier. Furthermore, we require performing one linear transform per border, where the operator matrix can have size \( \lambda \times 2\lambda \) in (14) or \( K \times K \) in (15), reflecting computational savings over [11], [12].

APPENDIX

To evaluate if \( G \) has indeed full-rank, regardless of the linear extension, let

\[
x_\infty^T = [\ldots x^T, (Ry)^T, (Rx)^T, (Rx)^T, \ldots] \quad (21)
\]

where \( R \) combines \( R_l \) and \( R_r \). In the worst case, we may impose that \( N_s \geq 2\lambda \). The signal represented by \( x_\infty \) is periodic with period \( 2N_s \) given by \( x_\infty^T = [x^T, (Rx)^T] \). Define \( T_x \) as \( T \) but with an infinite number of block-rows. Hence, \( x_\infty = T_x x \), is also periodic composed by vectors of the form \( y_\text{per} = [y^T, y^T] \), where nothing is said about the relation between \( y \) and \( y_\text{per} \). There exists a linear relation between \( y_\text{per} \) and \( x_\text{per} \) such that \( y_\text{per} = T_{\text{per}} x_\text{per} \), where \( T_{\text{per}} \) is an orthogonal block circulant matrix. [3] Rewriting this relation, dividing \( T_{\text{per}} \) into four \( N_s \times N_s \) submatrices, we have

\[
\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \quad (22)
\]

Then

\[
y = [T_{00} + T_{01} R] x = G x. \quad (23)
\]

Since \( x \) can be any vector in \( \mathbb{R}^{N_s} \), all possible combinations of elements of \( x_\text{per} \) span a subspace of \( \mathbb{R}^{N_s} \) of dimension \( N_s \). As \( T_{\text{per}} \) is orthogonal, rank \( \{[T_{00}, T_{10}]\} = N_s \) (full rank), and from (22), all possible combination of elements of \( y \) span \( \mathbb{R}^{N_s} \). Therefore, as the same applies for \( x \), we see from (23) that \( G \) has full rank and is one-to-one and onto. Thus, its inverse is unique, concluding the demonstration.

REFERENCES

that the $x_i$ are independent and identically distributed (i.i.d.) (e.g., [1], [4], [7]). In particular, an efficient method along the lines of inverse filtering has been proposed ([9]–[11]) that explicitly utilizes the discreteness of $\{x_i\}$ yet does not require the stationarity or other statistical information of $\{x_i\}$.

In this correspondence, we deal with the blind restoration problem under a Bayesian framework and by Gibbs sampling. The Gibbs sampling has been successfully applied to the ordinary image restoration problem by Geman and Geman [6] under the assumption that the filter $\{\psi_i\}$ and the statistical parameters of $\{x_i\}$ and $\{e_i\}$ are all available. In the present correspondence, we include these parameters in the list of unknowns and estimate them simultaneously with the signal $\{x_i\}$.

II. FORMULATION OF THE PROBLEM

Assume that the signal $\{x_i\}$ in (1) is a stationary first-order Markov chain with known state space $A := \{a_1, \ldots, a_n\}$ but unknown initial probabilities $\theta_i := pr(x_1 = a_i)$ and unknown transition probabilities $\theta_{ij} := pr(x_i = a_j | x_{i-1} = a_i)$. It is clear that the probabilities should satisfy the constraints $i = 1$ and $\sum_{j=1}^n \theta_{ij} = 1$ for $i = 1, \ldots, k$. Let $\theta$ denote the collection of these probabilities, namely $\theta := \{\theta_i, \theta_{ij} : i, j = 1, \ldots, k\}$. Although extensions to higher order Markov chains are quite straightforward, we restrict our effort to the first-order case for the simplicity of presentation. Assume further that $\{x_i\}$ in (1) is Gaussian white noise with zero-mean and unknown variance $\sigma^2$ and is independent of $\{e_i\}$.

Under these assumptions, the main objective of this correspondence is to simultaneously reconstruct the signal $x := \{x_1, \ldots, x_n\}$ and estimate the FIR filter $\phi := \{a_0, \ldots, a_{k-1}\}$ along with the statistical parameters $\sigma^2$ and $\theta$ on the basis of the data record $y := \{y_1, \ldots, y_n\}$. Note that the values $x_1, \ldots, x_0$ (that are outside the observation interval) are also included in $x$ for reconstruction and that the filter $\phi$ can be minimum phase or nonminimum phase. Noncausal FIR filters can be accommodated into the problem by a transformation of time index.

III. BAYESIAN APPROACH

The problem is solved under a Bayesian framework: First, the unknown quantities $x$, $\phi$, $\sigma^2$, and $\theta$ are regarded as realizations of random variables with suitable prior distributions. The Gibbs sampler, a Monte Carlo method, is then employed to calculate the minimum mean-squared error (MMSE) estimates and/or the maximum a posteriori (MAP) estimates of the unknowns.

A. Prior Distributions

In principle, prior distributions are used to incorporate our knowledge of the parameters, and less restrictive (or less informative) priors should be employed when such knowledge is limited. Computational complexity is another consideration that affects the selection. Conjugate priors are usually used to obtain simple analytical forms for the resulting posterior distributions (e.g., [2]). To make the Gibbs sampler more computationally efficient, the priors should also be chosen such that the conditional posterior distributions, as we shall see next, are easy to simulate.

For the restoration problem described above, the following priors are used in our procedure: to the filter $\phi$, we impose a Gaussian distribution $p(\phi) \sim N(\phi_0, \Sigma_0)$, and to the noise variance $\sigma^2$ we impose an inverted chi-square distribution $p(\sigma^2) \sim \chi^{-2}(\nu; \lambda)$, i.e., $\nu/\sigma^2 \sim \chi^2(\nu)$. Note that large values of $\Sigma_0$ and small values of $\nu$ correspond to less informative priors. Further, we use independent Dirichlet distributions as priors of $\theta_i$ and $\theta_{ij}$.