Approximating Lapped Transforms Through Unitary Postprocessing

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***Abstract*—In this letter, we deal with the approximation of a nonsquare lapped transform matrix by another matrix. This second matrix is constrained to be the product of two stages: one lapped transform and a unitary postprocessing step, in such a way that approximation is accomplished by modifying the postprocessing stage alone. Viewing both matrices as transforms, the goal is to minimize the variance of the output error (discrepancy) for given input signal statistics. An extension of the result to frequency responses of filter bank transfer matrices is also given along with an example to demonstrate the feasibility of the method. The example demonstrates the potential use of the method in modifying transforms which can be applied in audio compression.**

***Index Terms*—Lapped transforms, unitary approximations.**

I. Introduction

In a previous paper [1], it was shown how to find the optimal approximation of a non unitary transform by a unitary one in the sense of minimizing the discrepancy of the transformed signal. Unlike the classical approach [2], which shows how to minimize the norm of the difference between said matrices, the work in [1] is based on the input signal's statistics. In fact, minimizing the error norm (discrepancy) between the output signals [1], and the classical method of minimizing the error norm between matrix entries [2], [3] are identical tasks when the input signal is decorrelated, e.g., white noise. In this letter, we are interested in transforms that are used to transform blocks of samples from a wide sense stationary signal with known statistics. As in [1], we are interested in minimizing the output error incurred by replacing one matrix by another, but unlike the previous work we deal here with nonsquare matrices, i.e., we want to approximate a lapped transform (LT) by another. In effect, the work here is an extension of the work presented in [1] and is a generalization of the subspace rotation method ([3, Section XII-4.1]) for statistical signals instead of approximating matrix entries. As in the case of [1], the present method becomes a subspace rotation method as in [3] if the input is white noise. The framework will be better explained next.

II. Using Unitary Postprocessing to Approximate Two Lapped Transforms

Let the zero-mean input signal $x(n)$ be transformed by the $N \times M$ matrix $\mathbf{A} (N \leq M)$ on a block-by-block basis as $\mathbf{y} = \mathbf{Ax}$, where $\mathbf{x}$ represents a block of M input samples. Assume the signal $\{x(n)\}$ to have an M $\times$ M auto-correlation matrix $\mathbf{R}_{xx}$. Let $\mathbf{B}$ be another N $\times$ M matrix.

Let us assume one wants to approximate $\mathbf{B}$ to $\mathbf{A}$, and that the only tool one has available is to left-multiply $\mathbf{B}$ by a unitary N $\times$ N matrix $\mathbf{C}$ so that $\mathbf{D} = \mathbf{CB}$ is as close as possible to $\mathbf{A}$. Indeed, in this framework, one searches for the best rotation of the bases of $\mathbf{B}$ to approximate those of $\mathbf{A}$, even though $\mathbf{B}$ and $\mathbf{A}$ might not span the same vector space and might even be orthogonal.

We define the approximation error metric as the discrepancy between the output of $\mathbf{D}$ and $\mathbf{A}$, i.e.,

$$\epsilon = \mathbf{y} - \mathbf{Dx} = (\mathbf{A} - \mathbf{CB})\mathbf{x}. \tag{1}$$

We want to find the "best" $\mathbf{C}$ to minimize the *energy* of the discrepancy signal.

Theorem 1: The unitary matrix $\mathbf{C}$, which makes $\mathbf{D}$ be the closest to the full rank matrix $\mathbf{A}$ in the sense of minimizing the distance $J = E[\epsilon^H \epsilon]$ (error energy), is given by $\mathbf{C} = \mathbf{Q}_1\mathbf{Q}_2$, where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are the unitary matrices derived from the singular value decomposition (SVD) of $\mathbf{\Phi} = \mathbf{AR}_{xx}\mathbf{B}^H$ as $\mathbf{\Phi} = \mathbf{Q}_1\mathbf{\Lambda}\mathbf{Q}_2$.

Proof: $J = E[\epsilon^H \epsilon] = E[\mathbf{x}^H(\mathbf{A} - \mathbf{CB})^H(\mathbf{A} - \mathbf{CB})\mathbf{x}]$. By exchanging the inner product by the trace of the outer product we have

$$\begin{aligned} J &= E[\mathrm{Tr}\{(\mathbf{A} - \mathbf{CB})\mathbf{xx}^H(\mathbf{A} - \mathbf{CB})^H\}] \\ &= \mathrm{Tr}\{(\mathbf{A} - \mathbf{CB})\mathbf{R}_{xx}(\mathbf{A} - \mathbf{CB})^H\} \\ &= \mathrm{Tr}\{\mathbf{AR}_{xx}\mathbf{A}^H + \mathbf{BR}_{xx}\mathbf{B}^H - \mathbf{\Phi}\mathbf{C}^H - \mathbf{C}\mathbf{\Phi}^H\} \end{aligned} \tag{2}$$

where we used the fact that $\mathrm{Tr}\{\mathbf{CBR}_{xx}\mathbf{B}^H\mathbf{C}^H\} = \mathrm{Tr}\{\mathbf{BR}_{xx}\mathbf{B}^H\mathbf{C}^H\mathbf{C} = \mathrm{Tr}\{\mathbf{BR}_{xx}\mathbf{B}^H\}$, and where $\mathbf{\Phi} = \mathbf{AR}_{xx}\mathbf{B}^H$. We also used the fact that $\mathbf{R}_{xx}$ is a Hermitian matrix. In (2), note that the first two terms are independent of $\mathbf{C}$. Since for a square matrix $\mathbf{\Psi}, \mathrm{Tr}\{\mathbf{\Psi} + \mathbf{\Psi}^H\} = 2\mathrm{Re}\{\mathrm{Tr}\{\mathbf{\Psi}\}\}$, we can see that the minimization of J is equivalent to the maximization of

$$J' = \mathrm{Re}\{\mathrm{Tr}\{\mathbf{\Phi}\mathbf{C}^H\}\}. \tag{3}$$

Let the SVD of $\mathbf{\Phi}$ be given by $\mathbf{\Phi} = \mathbf{Q}_1\mathbf{\Lambda}\mathbf{Q}_2$, where the $\mathbf{Q}_i$ matrices are unitary and $\mathbf{\Lambda}$ is a diagonal matrix containing the nonnegative singular values of $\mathbf{\Phi}$. Then

$$\mathrm{Tr}\{\mathbf{\Phi}\mathbf{C}^H\} = \mathrm{Tr}\{\mathbf{Q}_1\mathbf{\Lambda}\mathbf{Q}_2\mathbf{C}^H\} = \mathrm{Tr}\{\mathbf{\Lambda}\mathbf{Q}_2\mathbf{C}^H\mathbf{Q}_1\}.$$

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Let $\mathbf{Q} = \mathbf{Q}_2\mathbf{C}^H\mathbf{Q}_1$ so that $\mathbf{Q}$ is some unitary matrix whose entries are denoted q_{ij}. Let the singular values of $\mathbf{\Phi}$ (diagonal entries in $\mathbf{\Lambda}$) be denoted by v_i. Hence

$$J' = \mathrm{Re}\left[\sum_{i=0}^{M-1} v_i q_{ii}\right]. \tag{4}$$

Furthermore, if $\mathbf{A}$ and $\mathbf{B}$ are full rank (rank N), so is $\mathbf{\Phi}$, and all v_i are positive and real. Therefore, since $\mathbf{Q}$ is unitary and $\sum_j |q_{ij}|^2 = 1$, (4) is maximized if and only if $q_{ii} = 1$ and $q_{ij} = 0$ for $i \neq j$, i.e., the cost is minimized if $\mathbf{Q} = \mathbf{I}_M$. As a result, $\mathbf{Q}_2\mathbf{C}^H\mathbf{Q}_1 = \mathbf{I}_M$ and $\mathbf{C}^H = \mathbf{Q}_2^H\mathbf{Q}_1^H$. Thus

$$\mathbf{C} = \mathbf{Q}_1\mathbf{Q}_2 \cdot \quad \triangle. \tag{5}$$

As was the case in [1], we can also derive three corollaries from this main theorem.

Corollary 1: If $\mathbf{A}$ or $\mathbf{B}$ are not full rank, a unitary matrix $\mathbf{C}$ that minimizes $E[\epsilon^T\epsilon]$, given in (5) is a nonunique solution. If all v_i are sorted by magnitude, the general solution is given by $\mathbf{C} = \mathbf{Q}_1\left[\begin{smallmatrix}\mathbf{I}_K & \mathbf{0}\\ \mathbf{0} & \mathbf{E}\end{smallmatrix}\right]\mathbf{Q}_2$, where $\mathrm{rank}(\mathbf{\Phi}) = K$ and $\mathbf{E}$ is any $(N-K)\times(N-K)$ unitary matrix.

Proof: If $\mathbf{A}$ or $\mathbf{B}$ do not have full rank, $\mathbf{\Phi}$ is not full rank either. Hence, some of the v_i are 0. If the rank of $\mathbf{\Phi}$ is K and if we sort the v_i in decreasing order, only the first $K v_i$ are nonzero. Then (4) becomes $\mathrm{Re}\{\sum_{i<K} v_i q_{ii}\}$ such that conditions $q_{ii} = 1$ and $q_{ij} = 0$ (for $i \neq j \,|\, \{j < K, \forall i\} \vee \{i < K, \forall j\}$) are necessary. However, for $i \geq K$, entries q_{ij} are irrelevant to (4). Thus, entries q_{ij} for $i \geq K$ and $j \geq K$ must form a unitary matrix from which $\mathbf{C} = \mathbf{Q}_1\mathbf{Q}^H\mathbf{Q}_2 = \mathbf{Q}_1\left[\begin{smallmatrix}\mathbf{I}_K & \mathbf{0}\\ \mathbf{0} & \mathbf{E}\end{smallmatrix}\right]\mathbf{Q}_2$ derives. As a result, $\mathbf{C} = \mathbf{Q}_1\mathbf{Q}_2$ is a nonunique closest approximation. $\triangle$

Corollary 2: For the case of a linear-lapped transform $\mathbf{A}$ with real entries and where $\{x(n)\}$ is a real signal, $\mathbf{C} = \mathbf{Q}_1\mathbf{Q}_2$ is a real orthogonal matrix that makes $\mathbf{D}$ approximate $\mathbf{A}$ in the sense of minimizing $E[\epsilon^T\epsilon]$.

Proof: In this case, $\mathbf{\Phi}$ is real and so are $\mathbf{Q}_1$ and $\mathbf{Q}_2$, thus $\mathbf{C}$ is orthogonal. $\triangle$.

We can also extend the result for some particular cases in polynomial matrices. Let $\mathbf{A}(z)$ be an N $\times$ M system made of N selected rows of the polyphase transfer matrix (PTM) of an M-channel filter bank. (See [4] for details of transfer matrices and filter banks.) The PTM relates the input vector $\mathbf{x}(z)$ containing the z-transforms $X_i(z)$ of the M polyphase components of $\{x(n)\}$ to the output vector $\mathbf{y}(z)$ which contains the z-transforms $Y_i(z)$ of the M subband signals, as $\mathbf{y}(z) = \mathbf{A}(z)\mathbf{x}(z)$. Assume $\{x(n)\}$ is periodic with a very large period N_p and let all Fourier transforms (FTs) be computed over one period. It can be shown that

$$E\left[X_i(e^{j\omega})X_j^H(e^{j\omega})\right] = N_p\Gamma_{ij}(e^{j\omega}) \tag{6}$$

where $\Gamma_{ij}(e^{j\omega})$ is the FT of the cross correlation between polyphases i and j, i.e., $\Gamma_{ij}(e^{j\omega}) = \mathcal{F}\{r_{xx}(m+i-j)\}$, where $\{r_{xx}(m)\}$ is the autocorrelation function of $\{x(n)\}$.

Let $\mathbf{B}(z)$ be another N $\times$ M system and $\mathbf{C}(z)$ be an N $\times$ N paraunitary matrix such that $\mathbf{D}(z) = \mathbf{C}(z)\mathbf{B}(z)$.

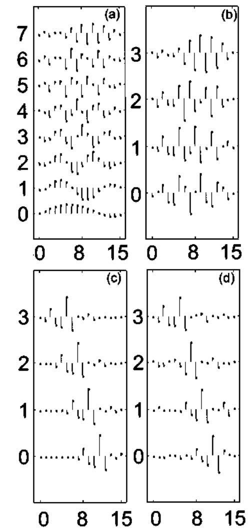

Fig. 1. Bases of the transforms. (a) Eight-channel MLT; (b) four higher frequency bases of the MLT used as $\mathbf{B}$; (c) desired matrix $\mathbf{A}$; and (d) resulting matrix $\mathbf{D}$.

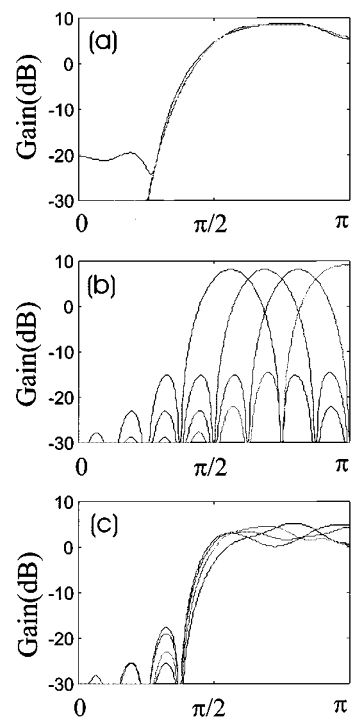

Fig. 2. Frequency responses of the filters associated with the bases (a) $\mathbf{A}$; (b) $\mathbf{B}$; and (c) $\mathbf{D}$.

Corollary 3: Let $\mathbf{\Gamma}(e^{j\omega})$ have entries $\Gamma_{ij}(e^{j\omega})$. The paraunitary PTM $\mathbf{C}(z)$, which makes $\mathbf{D}(z)$ be the closest to $\mathbf{A}(z)$ in the sense of minimizing the distance $J = E[\epsilon^H(e^{j\omega})\epsilon(e^{j\omega})]$, where $\epsilon(e^{j\omega}) = \mathbf{y}(e^{j\omega}) - \mathbf{D}(e^{j\omega})\mathbf{x}(e^{j\omega})$ has its frequency response governed by

$$\mathbf{C}(e^{j\omega}) = \mathbf{Q}_1(e^{j\omega})\mathbf{Q}_2(e^{j\omega}) \tag{7}$$

where $\mathbf{Q}_1(e^{j\omega})$ and $\mathbf{Q}_2(e^{j\omega})$ are unitary matrices derived from the SVD of $\mathbf{\Phi}(e^{j\omega}) = \mathbf{A}(e^{j\omega})\mathbf{\Gamma}(e^{j\omega})\mathbf{B}^H(e^{j\omega})$ as

$$\mathbf{\Phi}(e^{j\omega}) = \mathbf{Q}_1(e^{j\omega})\mathbf{\Lambda}(e^{j\omega})\mathbf{Q}_2(e^{j\omega}). \tag{8}$$

Proof: From (6), we have $E[\mathbf{x}(e^{j\omega})\mathbf{x}^H(e^{j\omega})] = N_p\mathbf{\Gamma}(e^{j\omega})$. Since N_p does not affect the minimization, the result in (7) follows from the proof of Theorem 1, where all the matrix entries are now a function of $(e^{j\omega})$, and $\mathbf{R}_{xx}$ is replaced by $\mathbf{\Gamma}(e^{j\omega})$. $\triangle$

Note that practical realizations of $\mathbf{C}(z)$ are compromised by the fact that the previous result does not mean that $\mathbf{C}(z)$ is even rational, i.e., it might not be realizable using a finite-order filter. If, instead, we had used a cost function integrating over ω, we would be led into filter bank design issues beyond the present scope.

III. Example

Consider a cosine-modulated filter bank [4], [5] such as those whose filters are in Fig. 1(a), which is an eight-channel bank of 16-tap filters. This is also known as the modulated LT (MLT). In some situations, it might be advantageous to shorten the effective length of the high-frequency bases, perhaps adaptively. This can be done using the proposed method. In an example, the four high frequency bases of the MLT are depicted in Fig. 1(b) and these are the bases composing our starting 4×8 matrix $\mathbf{B}$. To increase time resolution, we can use a reference matrix $\mathbf{A}$ whose bases are shown in Fig. 1(c). This matrix is a "desired state" and was obtained by summing the bases in $\mathbf{B}$ and creating its "shifted" versions to be used as rows of $\mathbf{A}$. Our choice was only an attempt to have $\mathbf{A}$ and $\mathbf{B}$ spanning similar spaces. The bases of the resulting matrix $\mathbf{D}$ are shown in Fig. 1(d), where the input signal was modeled as an AR(1) process with correlation coefficient $\rho = 0.95$. Note the improved space localization of the bases compared to those in Fig. 1(b). The frequency responses of the filters associated with $\mathbf{A}, \mathbf{B}$ and $\mathbf{D}$ are shown in Fig. 2.

In terms of applications, one idea is to adaptively switch between $\mathbf{B}$ and $\mathbf{D}$ on-the-fly by turning $\mathbf{C}$ on and off whenever convenient. For example, if one uses the MLT in audio compression [6], we can turn $\mathbf{C}$ on only when we desire to reduce pre-echo. The exploration of the method for audio compression is beyond the scope of the present paper. This application is the subject of ongoing work and a more detailed analysis will be presented in future work.

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