On Symmetric Extensions, Orthogonal Transforms of Images, and Paraunitary Filter Banks

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Abstract

Periodic or symmetric extensions are commonly used for processing images and other finite-length signals with a paraunitary filter bank (PUFB). Unlike infinite-length signals, PUFBs applied to finite-length signals will not necessarily lead to an orthogonal system. We show that for symmetric extensions, orthogonality is only possible for special PUFBs based on linear-phase filters. We also discuss implementation issues.

I Introduction

Multirate filter banks [1] are becoming increasingly popular for processing finite-length signals like images. Also, techniques for eliminating the border distortions caused by the discontinuity of the signal have been examined in different ways. The only trivial solution is to use periodic extensions and circular convolution, where operations can also be made with the aid of the discrete Fourier transform (DFT) [2]. However, the idea of assuming a periodic signal (see Fig. 1) implies that the samples in opposite borders of the image are adjacent and introduces artificial discontinuities. In image processing, the border distortions are generally easily identified because they have a well-defined space localization pattern, following the image boundary contours. Other sources of visible distortion patterns can arise from the use of improper reconstruction methods for the image boundaries, or from using schemes leading to non-orthogonal boundary filter banks, in which case, perfect reconstruction (PR) is assured, but the addition of near-white noise to the subbands (such as in quantization) leads to colored noise in the reconstructed signal at the boundaries. One simple approach that avoids artificial discontinuities is to assume a symmetric periodic signal [3]–[6], in the so-called symmetric extension method. We assume separable processing and study a finite-length 1D signal. In terms of notation, our conventions are: unidimensional concatenation of matrices and vectors is indicated by a comma; $[\,]^T$ means transposition; $\mathbf{I}_n$ is the $n\times n$ identity matrix; $\mathbf{0}_n$ is the $n\times n$ null matrix; and $\mathbf{J}_n$ is the $n\times n$ counter-identity matrix as $\mathbf{J}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

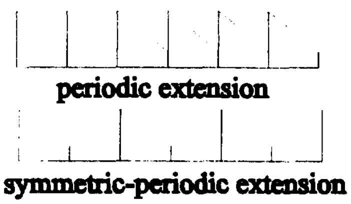

Figure 1: Illustration of signal extensions

II Paraunitary Filter Banks

We will use a PR critically-decimated paraunitary uniform filter bank (PUFB) [1] of M FIR filters. The filters are assumed to have a maximum length $L = NM$, and the analysis and synthesis filters have impulse reponses $f_k(n)$ and $g_k(n)$ (k=0,1,...,M-1, n=0,1,...,L-1), respectively. In a PUFB, $f_k(n) = g_k(L-1-n)$. The input signal, $x(n)$ is, thus, transformed by the analysis filter bank into the subband signals $y_k(m)$, which are processed yielding the samples $\hat{y}_k(m)$, and transformed by the synthesis filter bank into the reconstructed signal $\hat{x}(n)$, which is a delayed replica of $x(n)$, if $\hat{y}_k(m) = y_k(m)$. We can also define a transform matrix $\mathbf{P}$ with elements p_{ij} as

$$p_{ij} = f_i(L-1-j) = g_i(j) \quad (1)$$

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for $0 \leq i \leq M-1$ and $0 \leq j \leq L-1$ [7]. We can express the input signal in its polyphase components $x_i(m) = x(mM+i)$ and define a signal $y(n)$ whose polyphase components are the subbands, such that $y_i(m) = y(mM+i)$. Then, the polyphase transfer matrix (PTM) $\mathbf{F}(z)$ [1] is the multi-input multi-output FIR transfer matrix relating the polyphase components of $y(n)$ (the subbands) and of $x(n)$. If $\mathbf{P}$ is segmented into N square matrices as $\mathbf{P} = [\mathbf{P}_0\ \mathbf{P}_1\ \cdots\ \mathbf{P}_{N-1}]$, and as the paraunitary PTM requires that $\mathbf{F}^{-1}(z) = \mathbf{F}^T(z^{-1})$ [1], we have the following relations:

$$\mathbf{G}(z) = z^{-(N-1)}\mathbf{F}^T(z^{-1}). \tag{2}$$

$$\mathbf{F}(z) = \sum_{i=0}^{N-1} z^{-i}\mathbf{P}_{N-1-i}\mathbf{J}_M \tag{3}$$

$$\sum_{i=0}^{N-1-l} \mathbf{P}_i\mathbf{P}_{i+l}^T = \sum_{i=0}^{N-1-l} \mathbf{P}_i^T\mathbf{P}_{i+l} = \delta(l)\mathbf{I}_M. \tag{4}$$

We will consider more carefully the PUFBs which can be parameterized using the symmetric delay factorization (SDF). Let

$$\Lambda(z) = \begin{bmatrix} z^{-1}\mathbf{I}_{M/2} & 0 \\ 0 & \mathbf{I}_{M/2} \end{bmatrix}. \tag{5}$$

The SDF of the PTM is given by

$$\mathbf{F}(z) = \mathbf{B}_0 \prod_{i=1}^{N-1} (\Lambda(z)\mathbf{B}_i) \tag{6}$$

where all stages $\mathbf{B}_i$ are allowed to be arbitrary $M \times M$ orthogonal matrices. The flow-graph for a SDF-PUFB is shown in Fig. 2. SDF is not very restrictive in practice, although M is required to be even. All 2-channel PUFBs plus most of the cosine-modulated filter banks, and all the linear-phase PUFBs (LPPUFB) belong to this class, among others [1],[7]–[11].

If the vectors $\mathbf{x}$ and $\mathbf{y}$ contain the signals $x(n)$ and $y(n)$, of unrestricted length, respectively, then the analysis and synthesis sections can be represented in matrix notation as

$$\mathbf{y} = \mathbf{H}\mathbf{x} \tag{7}$$

$$\hat{\mathbf{x}} = \mathbf{H}^T\hat{\mathbf{y}} \tag{8}$$

where

$$\mathbf{H} = \begin{bmatrix} \ddots & \ddots & & \ddots & & & 0 \\ & \mathbf{P}_0 & \mathbf{P}_1 & \cdots & \mathbf{P}_{N-1} & & \\ & & \mathbf{P}_0 & \mathbf{P}_1 & \cdots & \mathbf{P}_{N-1} & \\ 0 & & & \ddots & \ddots & & \ddots \end{bmatrix}. \tag{9}$$

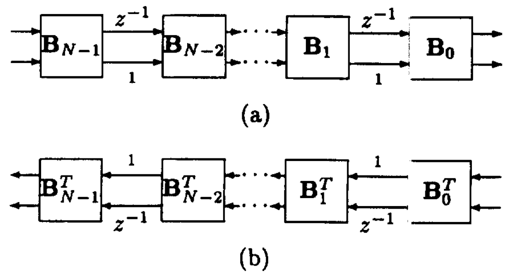

Figure 2: Flow graph for SDF-PUFB and N stages. All branches carry $M/2$ samples, and blocks $\mathbf{B}_i$ are $M \times M$ orthogonal matrices. (a) Analysis section; (b) Synthesis section.

Note that, from (4), $\mathbf{H}\mathbf{H}^T = \mathbf{H}^T\mathbf{H} = \mathbf{I}_\infty$, so that the transform mapping $\mathbf{x}$ into $\mathbf{y}$ is orthogonal.

Suppose the signal $x(n)$ has only N_x samples and assume $N_x = N_B M$, where N_B is an integer representing the number of blocks, with M samples per block. To avoid the expansion of the number of samples, we require $y(n)$ to have N_x samples, so that each subband would have N_B samples. As $\mathbf{x}$ and $\mathbf{y}$ are finite and $\mathbf{P}$ is not square, we may first extend $\mathbf{x}$ into the augmented vector $\tilde{\mathbf{x}}$, prior to the analysis process. Regardless of the extension method used, there is a size-limited linear transform $\mathbf{T}$ leading $\mathbf{x}$ into $\mathbf{y}$ so that [12]

$$\mathbf{y} = \mathbf{T}\mathbf{x} \tag{10}$$

and, if $\mathbf{T}$ is invertible,

$$\mathbf{x} = \mathbf{T}^{-1}\mathbf{y}. \tag{11}$$

We are seeking extensions methods and filter banks such that $\mathbf{T}$ is orthogonal.

III Periodic extensions

If, starting from $\mathbf{x}$, we create an infinite periodic signal (see Fig. 1) $\mathbf{x}_\infty$ as

$$\mathbf{x}_\infty^T = [\cdots, \mathbf{x}^T, \mathbf{x}^T, \mathbf{x}^T, \mathbf{x}^T, \mathbf{x}^T, \cdots], \tag{12}$$

then, we use (7) through (9) applied to signals $\mathbf{x}_\infty$ and $\mathbf{y}_\infty$ where $\mathbf{y}_\infty$ is also periodic given by

$$\mathbf{y}_\infty^T = [\cdots, \mathbf{y}^T, \mathbf{y}^T, \mathbf{y}^T, \mathbf{y}^T, \mathbf{y}^T, \cdots], \tag{13}$$

Hence $\mathbf{T}$ is block circulant and orthogonal [13]. As an example, for $N = 3$ and $N_B = 5$, we have

$$\mathbf{T} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{0}_M & \mathbf{0}_M & \mathbf{P}_0 \\ \mathbf{P}_0 & \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{0}_M & \mathbf{0}_M \\ \mathbf{0}_M & \mathbf{P}_0 & \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{0}_M \\ \mathbf{0}_M & \mathbf{0}_M & \mathbf{P}_0 & \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{P}_2 & \mathbf{0}_M & \mathbf{0}_M & \mathbf{P}_0 & \mathbf{P}_1 \end{bmatrix}$$

One can artificially produce an unlimited length signal, and apply a time-invariant filter bank to the extended signal. The equivalent size-limited transform $\mathbf{T}$, however, can be viewed as a time-varying filter bank, where the filter bank is changed near the signal boundaries to cope with the signal discontinuity. Let $N < N_B$ and $\mathbf{x}_r = \mathbf{J}_{N_x}\mathbf{x}$. We construct an infinite-length symmetric-periodic signal as

$$\mathbf{x}_\infty^T = [\cdots, \mathbf{x}^T, \mathbf{x}_r^T, \mathbf{x}^T, \mathbf{x}_r^T, \mathbf{x}^T, \mathbf{x}_r^T, \cdots]. \quad (14)$$

$\mathbf{x}_\infty$ is periodic and is processed by $\mathbf{T}_\infty$, as in (9), so that $\mathbf{y}_\infty = \mathbf{T}_\infty \mathbf{x}_\infty$, and, since both $\mathbf{T}_\infty$ and $\mathbf{x}_\infty$ have a periodic structure, then $\mathbf{y}_\infty$ is also periodic as

$$\mathbf{y}_\infty^T = [\cdots, \mathbf{y}^T, \mathbf{y}_r^T, \mathbf{y}^T, \mathbf{y}_r^T, \mathbf{y}^T, \mathbf{y}_r^T, \cdots], \quad (15)$$

where $\mathbf{y} = \mathbf{T}\mathbf{x}$ and $\mathbf{y}_r = \mathbf{E}_S\mathbf{y}$. One can show that [15], regardless of the linear extension, $\mathbf{T}$ is always invertible and that $\mathbf{E}_S$ always exists and is linear, being found by a simple formula.

Let the period of $\mathbf{x}_\infty$ and of $\mathbf{y}_\infty$ be represented by $\mathbf{x}_p^T = [\mathbf{x}^T, \mathbf{x}_r^T]$ and $\mathbf{y}_p^T = [\mathbf{y}^T, \mathbf{y}_r^T]$, respectively. The transform for one period is $\mathbf{T}_p$, where

$$\mathbf{y}_p = \mathbf{T}_p \mathbf{x}_p. \quad (16)$$

IV ORTHOGONALITY

Let $\mathbf{S}_{pre}$ and $\mathbf{S}_{pos}$ be orthogonal block diagonal matrices. If $\mathbf{T}$ is a size-limited transform based on $\mathbf{P}$, it is clear that $\mathbf{T}' = \mathbf{S}_{pos}\mathbf{T}\mathbf{S}_{pre}$ is also an orthogonal transform generated by a different filter bank, found by pre- or post-processing the filter bank input or output with trivial block-transform operations. Except for pre- or post-processing, symmetric extensions will lead to an orthogonal size-limited transform $\mathbf{T}$ if and only if the filters in the SDF-PUFB have linear-phase.

As we discussed earlier, $\mathbf{T}_p$ is a block circulant orthogonal matrix [13]. Note that $\mathbf{T}_p$ can be divided into four $N_x \times N_x$ square submatrices as

$$\mathbf{T}_p = \begin{bmatrix} \mathbf{T}_0 & \mathbf{T}_1 \\ \mathbf{T}_1 & \mathbf{T}_0 \end{bmatrix}. \quad (17)$$

From $\mathbf{T}_p^T\mathbf{T}_p = \mathbf{T}_p\mathbf{T}_p^T = \mathbf{I}_{2N_x}$, and from (17), we obtain the following relations:

$$\begin{aligned} \mathbf{T}_0\mathbf{T}_0^T + \mathbf{T}_1\mathbf{T}_1^T &= \mathbf{I}_{N_x} \\ \mathbf{T}_0\mathbf{T}_1^T + \mathbf{T}_1\mathbf{T}_0^T &= \mathbf{0}_{N_x} \\ \mathbf{T}_0^T\mathbf{T}_0 + \mathbf{T}_1^T\mathbf{T}_1 &= \mathbf{I}_{N_x} \\ \mathbf{T}_0^T\mathbf{T}_1 + \mathbf{T}_1^T\mathbf{T}_0 &= \mathbf{0}_{N_x} \end{aligned} \quad (18)$$

Consider a linear-phase filter bank and define a $M \times M$ diagonal matrix $\tilde{\mathbf{V}}$ with elements $v_{kk} = 1$, if $f_k(m)$ is symmetric and $v_{kk} = -1$, if $f_k(m)$ is anti-symmetric. Then,

$$\mathbf{P} = \tilde{\mathbf{V}}\mathbf{P}\mathbf{J}_L. \quad (19)$$

Let $\mathbf{V}$ be an $N_x \times N_x$ matrix with non-zero block entries only in the counter-diagonal, as

$$\mathbf{V} = \begin{bmatrix} 0 & & & \tilde{\mathbf{V}} \\ & & \tilde{\mathbf{V}} & \\ & \tilde{\mathbf{V}} & & \\ \tilde{\mathbf{V}} & & & 0 \end{bmatrix} \quad (20)$$

Using (19), it is easy to verify that

$$\begin{aligned} \mathbf{T}_0 &= \mathbf{V}\mathbf{T}_0\mathbf{J}_{N_x} \\ \mathbf{T}_1 &= \mathbf{V}\mathbf{T}_1\mathbf{J}_{N_x}. \end{aligned} \quad (21)$$

Substituting (21) into (18), and using the fact that $\mathbf{V}$ is orthogonal, we get

$$\begin{aligned} \mathbf{T}_0\mathbf{T}_0^T + \mathbf{T}_1\mathbf{T}_1^T &= \mathbf{I}_{N_x} \\ \mathbf{T}_0\mathbf{J}_{N_x}\mathbf{T}_1^T + \mathbf{T}_1\mathbf{J}_{N_x}\mathbf{T}_0^T &= \mathbf{0}_{N_x} \\ \mathbf{T}_0^T\mathbf{T}_0 + \mathbf{J}_{N_x}\mathbf{T}_1^T\mathbf{T}_1\mathbf{J}_{N_x} &= \mathbf{I}_{N_x} \\ \mathbf{T}_0^T\mathbf{T}_1\mathbf{J}_{N_x} + \mathbf{J}_{N_x}\mathbf{T}_1^T\mathbf{T}_0 &= \mathbf{0}_{N_x}. \end{aligned} \quad (22)$$

From (16) and (17), we have $\mathbf{y} = (\mathbf{T}_0 + \mathbf{T}_1\mathbf{J}_{N_x})\mathbf{x}$, so that $\mathbf{T} = \mathbf{T}_0 + \mathbf{T}_1\mathbf{J}_{N_x}$. This result combined with (22) can be used to check that $\mathbf{T}\mathbf{T}^T = \mathbf{T}^T\mathbf{T} = \mathbf{I}_{N_x}$ and to prove the sufficiency of the statement. To prove the necessity, note that given (18) we would reach (22) by algebraic manipulation if and only if we have

$$\begin{aligned} \mathbf{T}_0 &= \mathbf{\Phi}'\mathbf{T}_0\mathbf{J}_{N_x} \\ \mathbf{T}_1 &= \mathbf{\Phi}'\mathbf{T}_1\mathbf{J}_{N_x}. \end{aligned} \quad (23)$$

where $\mathbf{\Phi}'$ is an orthogonal matrix. As $\mathbf{T}_0$ and $\mathbf{T}_1$ are block-circulant, presenting a periodic structure, the reader can check that (23) is only possible if $\mathbf{P}$ presents a structure such that

$$\mathbf{P} = \mathbf{\Phi}\mathbf{P}\mathbf{J}_L, \quad (24)$$

where $\mathbf{\Phi}$ is a square orthogonal matrix. In other words, the filters $f_k(n)$ would have to be found by a linear combination of their time-reversed versions (which are the filters $g_k(n)$). Using the fact that $\mathbf{P}\mathbf{P}^T = \mathbf{I}_M$ and after some manipulation, we can see that (24) is only true if $\mathbf{\Phi} = \mathbf{\Phi}^{-1}$ ($\mathbf{\Phi}$ is symmetric and orthogonal) and that $\mathbf{\Phi} = \mathbf{P}\mathbf{J}_L\mathbf{P}^T$. For such a matrix there is an orthogonal matrix $\mathbf{A}$, such that $\mathbf{D} = \mathbf{A}^T\mathbf{\Phi}\mathbf{A}$ [14], where

$\mathbf{D}$ is a diagonal matrix. As $\mathbf{A}$ and Φ are orthogonal, $\mathbf{D}$ is orthogonal, having elements ± 1 along the diagonal. Therefore, for every solution, $\mathbf{P}$, to (24), there is a solution, $\mathbf{P}_{LP}$, corresponding to linear-phase filters, where $\mathbf{P}_{LP} = \mathbf{A}^T\mathbf{P}$, such that $\mathbf{P}_{LP} = \mathbf{D}\mathbf{P}_{LP}\mathbf{J}_L$ and $\mathbf{P}_{LP}\mathbf{J}_L\mathbf{P}_{LP}^T = \mathbf{D}$. So, every solution, $\mathbf{P}$, to (24) can be written as $\mathbf{P} = \mathbf{A}\mathbf{P}_{LP}$, which means a post-processing of a LPPUFB by a block-transform $\mathbf{A}$. This concludes the proof.

We conclude that we can always ensure PR and orthogonality using symmetric extensions for LPPUFB. Furthermore, non-linear-phase filters cannot achieve orthogonality using a symmetric extension (except for the trivial case discussed).

V Implementation of LPPUFB

For the symmetric extension, it is easy to see that

$$\begin{aligned} \mathbf{y}_r &= \mathbf{T}_1\mathbf{x} + \mathbf{T}_0\mathbf{J}_{N_x}\mathbf{x} = (\mathbf{V}\mathbf{T}_1\mathbf{J}_{N_x} + \mathbf{V}\mathbf{T}_0)\mathbf{x} \\ &= \mathbf{V}(\mathbf{T}_0 + \mathbf{T}_1\mathbf{J}_{N_x})\mathbf{x} = \mathbf{V}\mathbf{T}\mathbf{x} = \mathbf{V}\mathbf{y} \end{aligned} \quad (25)$$

and as a result, using symmetric extensions, $\mathbf{y}_r$ is easily found from $\mathbf{y}$ by sample mirroring (for each subband) and sign inversions.

V.1 Time-domain Implementation

In Fig. 2, we have a *clocked* system with memory where at each instant (block index) a block of M samples in time-domain is the input which is tranformed into another block of M subband samples. Based on the previous results, for the analysis, we extend the signal, through a mirror-image reflection applied to the last $\lambda = (L-M)/2$ samples on each border, resulting in a vector $\tilde{\mathbf{x}}$ with $N_x + 2\lambda$ samples. Let $\mathbf{u}_1^T = [x(\lambda-1), \ldots, x(1), x(0)]$ and $\mathbf{u}_2^T = [x(N_x-1), x(N_x-2), \ldots, x(N_x-\lambda)]$. Then

$$\tilde{\mathbf{x}}^T = [\mathbf{u}_1^T, \mathbf{x}^T, \mathbf{u}_2^T]$$

The internal states in Fig. 2(a) can be initialized in any fashion and the signal is processed yielding $N_B + N - 1$ blocks. We discard the first $N-1$ output blocks, obtaining N_B transform-domain blocks corresponding to N_B samples of each subband.

At the synthesis section, we have the subband signals $\hat{y}_k(m)$ composing the signal $\hat{y}(n)$ as $\hat{y}(mM+i) = \hat{y}_i(m)$ for $0 \le i \le M-1$. Let K be the integer resulting from the integer division $N/2$. This signal $\hat{y}(n)$ is extended, by extending the subband signals by K samples in each border, as in (25), and processed as in Fig. 2(b). Let $\mathbf{v}_k^T = v_{kk}[\hat{y}_k(K-1), \cdots, \hat{y}_k(0)]$, $\mathbf{w}_k^T = v_{kk}[\hat{y}_k(N_B-1), \cdots, \hat{y}_k(N_B-K)]$. The k-th subband (initially having N_B samples and denoted by vector $\mathbf{s}_k$) is extended as

$$[\mathbf{v}_k^T, \mathbf{s}_k^T, \mathbf{w}_k^T]$$

Then, we proceed with the synthesis over the $N_B + 2K$ blocks of $\hat{y}(n)$, obtaining a reconstructed signal with $N_B + 2K$ blocks $\hat{\tilde{x}}(n)$, initializing the states of Fig. 2(b) in any fashion. For N odd, $K = (N-1)/2$, we discard the first $N-1$ blocks to obtain $\hat{x}(n)$. For N even ($K = N/2$), we discard the first $N-1$ blocks, the first $M/2$ samples in the N-th block and the last $M/2$ samples of the signal.

In the absence of quantization/processing of the subbands, $\hat{x}(n) = x(n)$. This approach will assure the perfect reconstruction property and orthogonality of the analysis and synthesis processes, paying the price of running the algorithm over extra N or $N-1$ blocks, making it suitable for applications where $N_B \gg N$.

V.2 DFT-aided Implementation

In some applications, it may be more convenient to implement the LPPUFB with the aid of the DFT. For this, the filtering/subsampling, or the upsampling/filtering operations can be performed in the DFT domain, as long as the signal is periodic. For the symmetric extension method, the periodic vectors are $\mathbf{x}_p$ and $\mathbf{y}_p$, and the transform $\mathbf{T}_p$ and its inverse $\mathbf{T}_p^T$ is the one implemented in the DFT-domain. Using (16) and (25), we have

$$\mathbf{y}_p = \begin{bmatrix} \mathbf{y} \\ \mathbf{V}\mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{N_x} \\ \mathbf{V} \end{bmatrix} \mathbf{y} \quad (26)$$

$$\mathbf{x}_p = \begin{bmatrix} \mathbf{x} \\ \mathbf{J}_{N_x}\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{N_x} \\ \mathbf{J}_{N_x} \end{bmatrix} \mathbf{x} \quad (27)$$

$$\mathbf{y} = \frac{1}{2} \begin{bmatrix} \mathbf{I}_{N_x} & \mathbf{V} \end{bmatrix} \mathbf{T}_p \begin{bmatrix} \mathbf{I}_{N_x} \\ \mathbf{J}_{N_x} \end{bmatrix} \mathbf{x} \quad (28)$$

$$\mathbf{x} = \frac{1}{2} \begin{bmatrix} \mathbf{I}_{N_x} & \mathbf{J}_{N_x} \end{bmatrix} \mathbf{T}_p \begin{bmatrix} \mathbf{I}_{N_x} \\ \mathbf{V} \end{bmatrix} \mathbf{y}. \quad (29)$$

We use a DFT of size $2N_x$ of a symmetric real sequence of length $2N_x$, reducing the DFT computation close to the complexity of a N_x-samples DFT. Filtering and subsampling is computed in the DFT domain followed by an inverse DFT, to whose output we apply N_x additions. For the synthesis, the procedure is similar, where the subbands are extended in a symmetric way, and upsampling followed by filtering are performed in the DFT domain. As in the subsampling case, we apply N_x additions to the output of the inverse DFT.

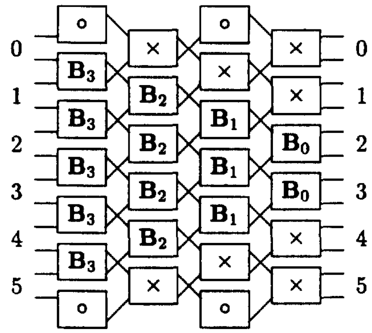

Figure 3: Flow-graph for a size-limited PUFB for $N = 4$ and $N_B = 6$. Each branch carries $M/2$ samples. The 6 input and output blocks are numbered and $\times$ stands for an $M \times M$ sparse factor, while $\circ$ is an $M/2 \times M/2$ factor.

VI Conclusions

We have shown that orthogonality of the boundary filter banks can be achieved with symmetric extensions and LPPUFBs. What if one decides to use a non-linear-phase PUFB? Perfect reconstruction is not possible using the methods described for LPPUFBs, because they assume orthogonality of the sparse factors. To obtain perfect reconstruction, one may waive either the symmetric extension or the orthogonality. If symmetric extensions are maintained, the sparse factors $\mathbf{B}_i$, for the size-limited transform, will be non-orthogonal and may be substituted by their inverses at the synthesis section. On the other hand, we can change the boundary filter banks in order to ensure full orthogonality of the size-limited transform $\mathbf{T}$ [16],[17]. Furthermore, it is also possible to obtain an optimal (for a given cost function) set of boundary filter banks. As an example, Fig. 3 shows a size-limited flow-graph to implement an $N = 4$ PUFB with non-linear-phase filters. The boundary factors can be constructed as to emulate a symmetric extension, in which case they will not be orthogonal. On the other hand they can be made orthogonal, thus ensuring full orthogonality of $\mathbf{T}$. Such approach is easily generalized and it actually employs the concepts of time-varying PUFBs. These methods are a continuation of this paper and their inclusion was precluded due to space limitations. See [15] and [18] for more on the subject.

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