

Optimal Orthogonal Boundary Filter Banks

Ricardo L. de Queiroz* K. R. Rao
 Xerox Co. Electrical Engineering Department
 800 Phillips Rd 128-27E University of Texas at Arlington
 Webster, NY, 14580 Box 19016, Arlington, TX, 76019
 queiroz@wrc.xerox.com rao@ee.uta.edu

Abstract - The use of a paraunitary filter bank for image processing requires a special treatment at image boundaries to ensure perfect reconstruction and orthogonality of these regions. Using time-varying boundary filter banks, we will discuss a procedure that explores all degrees of freedom of the border filters in a method essentially independent of signal extensions, allowing us to design optimal boundary filter banks, while maintaining fast implementation algorithms.

I INTRODUCTION

The applications of multirate filter banks [1] in image processing are receiving increasing attention and so are the problems resulting from the processing of finite length signals. This work is a continuation of the one in [2], and here we explore the concept of time-varying orthogonal filter banks so that filter banks near the borders are changed to force overall orthogonality [3]–[8].

We use a PR critically-decimated paraunitary uniform filter bank [1] of M FIR filters. The filters are assumed to have a maximum length $L = NM$, where N is also called the overlap factor, and the analysis and synthesis filters have impulse responses $f_k(n)$ and $g_k(n)$ ($k=0,1,\dots,M-1$, $n=0,1,\dots,L-1$), respectively. We will refer to such system shortly as a paraunitary filter bank (PUFB). The input signal, $x(n)$ is, thus, transformed by the analysis filter bank into the subband signals $y_k(m)$. In a PUFB, we can define a lapped transform matrix \mathbf{P} with elements p_{ij} as

$$p_{ij} = f_i(L - 1 - j) = g_i(j), \quad (1)$$

and \mathbf{P} can be segmented into N square matrices as

$$\mathbf{P} = [\mathbf{P}_0 \ \mathbf{P}_1 \ \cdots \ \mathbf{P}_{N-1}], \quad (2)$$

*The author was with Univ. of Texas at Arlington. This work was supported in part by CNPq, Brazil, under Grant 200.804-90-1.

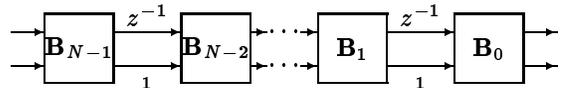


Figure 1: Flow graph for the analysis section of a paraunitary FIR filter bank where $\mathbf{E}(z)$ can be factorized using symmetric delays and N stages.

Let the input signal $x(n)$ be expressed in its polyphase components $x_i(m) = x(mM + i)$ and define a signal $y(n)$ whose polyphase components are the subbands, such that $y_i(m) = y(mM + i)$. Thus, the two sets of M signals are related by a polyphase transfer matrix (PTM) $\mathbf{E}(z)$ [1] which is paraunitary, i.e. $\mathbf{E}^{-1}(z) = \mathbf{E}^T(z^{-1})$. We will consider the PUFBs which can be parameterized using the symmetric delay factorization (SDF). Let

$$\Lambda(z) = \begin{bmatrix} z^{-1}\mathbf{I}_{M/2} & 0 \\ 0 & \mathbf{I}_{M/2} \end{bmatrix}. \quad (3)$$

The SDF of the PTM is given by

$$\mathbf{E}(z) = \mathbf{B}_0 \prod_{i=1}^{N-1} (\Lambda(z)\mathbf{B}_i) \quad (4)$$

where all stages \mathbf{B}_i are allowed to be arbitrary $M \times M$ orthogonal matrices. The flow graph for implementing a PUFB which can be parameterized using SDF is shown in Fig. 1. The use of SDF is not very restrictive in practice, as, for M even, most PUFBs with any practical advantage can be expressed in this way. Examples of such filter banks are linear-phase PUFBs [10], cosine modulated filter banks [1, 9], and, of course, all 2-channel PUFB. Its great advantage is that it spares us the task of developing different algorithms for each border of the signal.

If the vectors \mathbf{x} and \mathbf{y} contain the signals $x(n)$ and $y(n)$, respectively, then the analysis and synthesis sections can be represented in matrix notation as

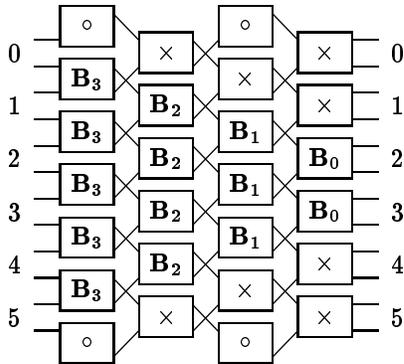


Figure 2: Flow-graph for a size-limited orthogonal implementation of a PUFB for $N = 4$ and $N_B = 6$. Each branch carries $M/2$ samples.

When the pruning process is complete, and the boxes belonging to the transitory boundary filter banks are selected, we will have some orthogonal matrices as degrees of freedom. An $n \times n$ orthogonal matrix has $n(n-1)/2$ degrees of freedom corresponding to its plane rotation angles. The reader can check that, for each border, the number of generic orthogonal boundary matrices is

stage $2i$: $K-i$ matrices of size $M \times M$

stage $2i+1$: $K-i-1$ matrices of size $M \times M$ and one of size $\frac{M}{2} \times \frac{M}{2}$

The number of degrees of freedom for each border is

$$\begin{aligned} \nu &= \left(\sum_{i=1}^K i + \sum_{i=1}^{K-1} i \right) \frac{M(M-1)}{2} + K \frac{\frac{M}{2}(\frac{M}{2}-1)}{2} \\ &= [(4K+1)(M-1) - 1] \frac{KM}{8} \end{aligned} \quad (11)$$

In the design of the boundary filter banks, for an optimal orthogonal solution we shall span all degrees of freedom in a search for the minimum of a specific cost function. As the relation among the plane rotations and cost functions is generally non-linear, an optimization algorithm would generally have slow convergence and lead to a local minimum. So, a large number of variables to optimize can be burdensome. Note that ν can be a very big number.

III DESIGN EXAMPLES

In a simple example, for $M = 2$, $N = 4$ and $N_B = 6$ (see Fig. 2) we have 4 degrees of freedom at each border. ($M/2 = 1$ and the 1×1 “orthogonal” matrices are set to 1.) We started with a 2-channel 8-tap PR PUFB shown in Fig. 3(a) and used an unconstrained non-linear optimization routine to optimize the border matrices (one plane rotation angle per matrix),

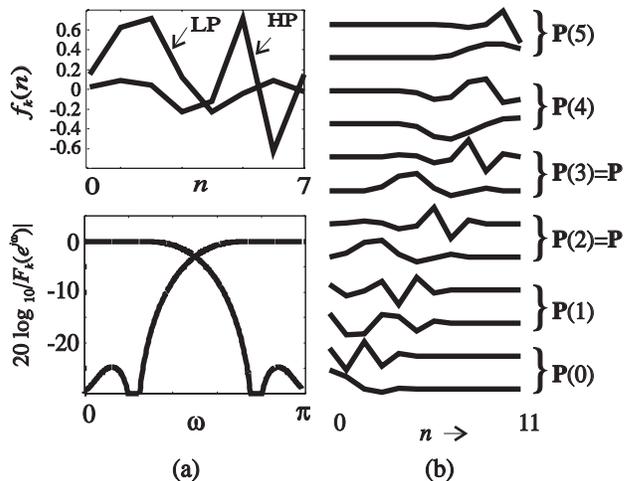


Figure 3: Design example of orthogonal boundary filter banks based on a 2-channel PUFB.

where the function maximized was an average of the stopband attenuation of the boundary filters. The 12 resulting bases for the 12-sample signal are shown in Fig. 3(b), where the relation of the basis functions and $\mathbf{P}(m)$ ($m = 0, \dots, 5$) is indicated. Note that $\mathbf{P}(2) = \mathbf{P}(3) = \mathbf{P}$ for MCD, and the 4 bases in the middle of Fig. 3(b) are the same as those in Fig. 3(a).

As a second example, more tuned to image coding applications, we used the modulated lapped transform (MLT) [9] with $N = 2$ and $M = 8$. We have 34 degrees of freedom at each border since just one filter bank needs to be optimized because $K = 1$. Malvar [9] provided a standard boundary solution for the MLT which is orthogonal and, therefore, it is a special case among all solutions wherein the 34 degrees of freedom per border would span. Here, we maximized the transform coding gain G_{TC} [11] for the boundary filter bank. Assuming $x(n)$ has autocorrelation $r_x(n) = 0.95^{|n|}$ and an autocorrelation matrix \mathbf{R}_{xx} with entries $R_{ij} = 0.95^{|i-j|}$ ($0 \leq (i, j) \leq L-1$), and denoting the diagonal elements of $\mathbf{P}\mathbf{R}_{xx}\mathbf{P}^T$ as σ_0^2 through σ_{M-1}^2 , then

$$G_{TC} = 10 \log_{10} \left(\frac{1}{M} \sum_{i=0}^{M-1} \sigma_i^2 \right) \left(\prod_{i=0}^{M-1} \sigma_i^2 \right)^{-1/M}. \quad (12)$$

In one border, the G_{TC} for the optimal boundary filter bank is 9.19dB, compared to 5.66dB from that of Malvar. As a reference, the MLT has G_{TC} ranging from 8.25dB through 9.22dB (it depends upon a design parameter [9]) and the popular discrete cosine transform (DCT) has $G_{TC} = 8.83$ dB.

We carried image coding tests, using the MLT, a 48×48 -pels image and adopting $M = 8$. For a two-dimensional separable implementation, there are

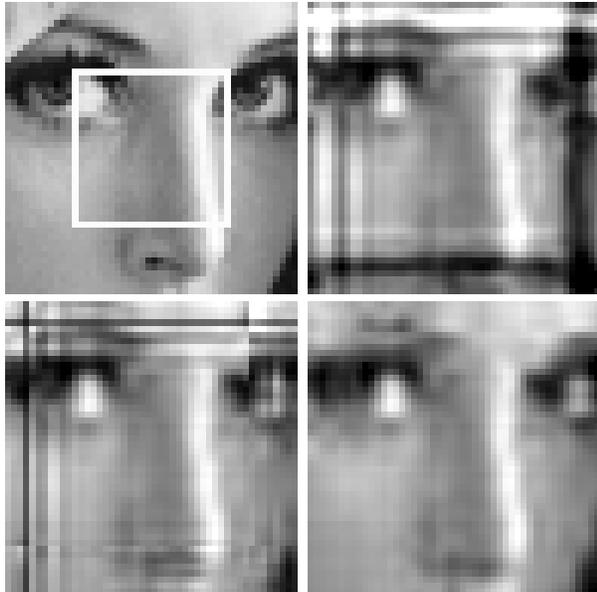


Figure 4: High-compression simulation with the $M = 8$ MLT. From top to bottom and left to right: original image; periodic extension; standard boundary MLT; boundary filter bank optimized for maximum G_{TC} .

$8^2 = 64$ subbands, and 36 coefficients in each subband, where 20 of them result from boundary filter banks. So, as each basis function (filter) has 16 elements ($N = 2$), boundary PUFBs will affect a 12-pixels-deep region of the reconstructed image starting at each border. We carried a comparison, using periodic extension (circular convolution), the standard boundary orthogonal solution proposed by Malvar [9] and our optimal solution (maximum G_{TC}). We quantized only 8 out of 64 subbands and the results are shown in Fig. 4, where it can be seen that the optimal boundary solution is free of border distortion at high-compression rates, while the other methods are not.

IV CONCLUSIONS

We have developed a technique to construct orthogonal boundary filter banks for PUFBs. The restrictions imposed are minimal, and the results can be changed to accommodate an odd number of channels M or a non-symmetric factorization, in which case the basic ideas would not change, but the presentation would be greatly complicated. Simplification, in fact, is the reason behind the choice for restricting the length of the signal N_x to be a multiple of M . The motivation for studying finite-length signals is for the application of PUFBs in image processing/coding. In this case, the image dimensions are often chosen as a multiple

of M , otherwise, the image is artificially extended, as do most of the image coders. Methods to construct orthogonal boundary filter banks have been reported earlier [4]–[8], however we have presented a general solution (as long as the PUFB obeys the SDF), and have explicitly pointed the degrees of freedom of such transitions. This allowed us to easily design optimal boundary filter banks. The absence of border distortion is also clear from our image coding tests using optimized boundary filter bank, providing a great improvement in relation to existing orthogonalization methods for the MLT.

A more complete discussion of the work in this paper and of [2] can be found in [5, 12].

REFERENCES

- [1] P.P. Vaidyanathan, *Multirate Systems and Filter Banks*. Englewood Cliffs, NJ: Prentice-Hall, 1993.
- [2] R. L. de Queiroz and K. R. Rao, "On symmetric extensions, orthogonal transforms of images, and paraunitary filter banks," *Proc. of IEEE Intl. Conf. on Image Processing*, Austin, TX, vol. I, pp. 835–839, 1994.
- [3] J. Pesquet, "Orthonormal wavelets for finite sequences," *Proc. of ICASSP*, vol. IV, pp. 609–612, 1992.
- [4] R.L. de Queiroz and K. R. Rao, "Time-varying lapped transforms and wavelet packets," *IEEE Trans. on Signal Processing*, vol. 41, pp. 3293–3305, Dec. 1993.
- [5] R.L. de Queiroz and K. R. Rao, "On orthogonal transforms of images using paraunitary filter banks," to appear at *Journal of Visual Communications and Image Representation*.
- [6] R. A. Gopinath and C. S. Burrus, "Factorization approach to unitary time-varying filter banks," preprint.
- [7] C. Herley, J. Kovacevic, K. Ramchandran and M. Vetterli, "Tilings of the time-frequency plane: construction of arbitrary orthogonal bases and fast tiling algorithms," *IEEE Trans. on Signal Processing*, vol. 41, pp. 3341–3359, Dec. 1993.
- [8] C. Herley and M. Vetterli, "Orthogonal time-varying filter banks and wavelets," *Proc. of ISCAS*, Chicago, IL, Vol. I, pp. 391–394, May 1993.
- [9] H. S. Malvar, *Signal Processing with Lapped Transforms*. Norwood, MA: Artech House, 1992.
- [10] A. K. Soman, P. P. Vaidyanathan, and T. Q. Nguyen, "Linear-phase paraunitary filter banks: theory, factorizations and applications," *IEEE Trans. on Signal Processing*, vol. 41, pp. 3480–3496, Dec. 1993.
- [11] N. S. Jayant and P. Noll, *Digital Coding of Waveforms*. Englewood Cliffs, NJ: Prentice-Hall, 1984.
- [12] R. L. de Queiroz, *On Lapped Transforms*, Ph.D. Dissertation. University of Texas at Arlington, Aug. 1994.