# ADAPTIVE EXTENDED LAPPED TRANSFORMS

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## Abstract

The perfect reconstruction conditions for a time-varying lapped transform are presented. The emphasis is on a class of cosine modulated filter banks known as the extended lapped transforms. A mechanism for achieving this error-free variation thru the factorization of the transform matrix into sparse factors is presented along with a discussion of transition states. Some possible applications are indicated, including a structure for perfect reconstruction orthogonal time-varying wavelet packets.

## 1 Introduction

Multirate filter banks [1] are well-known powerful tools in modern digital signal processing allowing easy data processing in transform domain, and flexible time-frequency analysis. The common denomination of "paraunitary filter bank" will be replaced here by the term "lapped transform". Although studied independently in the past, both represent the same concept [3]. Filter banks are generally thought of as stationary forms. Recently, Nayebi et al. presented a study on the structure of time-varying filter banks [4]. In their work, perfect reconstruction (PR) conditions were stated and it was mainly focused on the transition process between two known PR systems. In this paper, we present a structure that is inherently orthogonal and is based on lapped transforms with fast algorithms. We consider the extended lapped transform (ELT) as defined by Malvar in [3, 5], among the most efficient factorization methods for a paraunitary filter bank. Although imposing restrictions (that lead to fast algorithms) the resulting filter bank presents very good frequency responses [3]. For this reason, we will present in detail the structure for an adaptive form of the ELT.

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## 2 The ELT

We are assuming the use of an uniform analysis bank of $M$ FIR filters, each one of length $L$ . $L$ is related to $M$ as $L = NM = 2KM$. $K$ will be also refered here as the overlapping factor. There is a property of lapped transforms that relates the $M$ analysis and synthesis filter banks, which states that the analysis filters are time-reversed versions of the synthesis filters [3]. If the analysis and synthesis filters are represented by $f_m(n)$ and $g_m(n)$, respectively, for $m = 0, 1, \ldots, M-1$ and $n = 0, 1, \ldots, L-1$, we can define a matrix $\mathbf{P}$ with elements $p_{mn}$ as $p_{mn} = g_m(n) = f_m(L-1-n)$ Note that $\mathbf{P}$ is a $M \times L$ matrix, which will throughout this paper be called the transform matrix. For an ELT, the filters' length is an even multiple of the blocksize ($L = 2KM$) and $\mathbf{P}$ can be factorized into $K+1$ stages as:

$$\mathbf{P} = \mathbf{Z}\,\mathbf{D}_0\,\mathbf{B}_0\,\mathbf{D}_1\,\mathbf{B}_1\,\cdots\,\mathbf{D}_{K-1}\,\mathbf{B}_{K-1} \qquad (1)$$

$\mathbf{Z}$ is an $M \times M$ DCT type IV matrix [6], with inverted inputs as

$$\mathbf{Z} = \mathbf{DCT}^{\mathrm{IV}} \begin{pmatrix} \mathbf{0}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & \mathbf{0}_{M/2} \end{pmatrix} \qquad (2)$$

The $\mathbf{D}_n$ matrices have dimensions: $M \times 2M$, for $n = 0$, and $2nM \times (2n+2)M$, for $1 \leq n \leq K-1$. These matrices are more easily described as block matrices generated by a Kronecker product.

$$\mathbf{D}_n = \mathbf{F}_n \otimes \mathbf{I}_{M/2} \qquad (3)$$

where $\otimes$ denotes the Kronecker product. The $\mathbf{F}_n$ matrices (with elements $f_{ij,n}$) are

$$\mathbf{F}_0 = \{f_{ij,0}\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (4)$$



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and $\mathbf{F}_n = \{f_{ij,n}\}$, where

$$f_{ij,n} = \begin{cases} 1 & i = j = 2k \\ 1 & i = 4n - 1 - 2k \text{ and } j = 4n + 3 - 2k \\ 0 & \text{otherwise} \\ \text{for} & 0 \le k \le 2n - 1, n > 0 \end{cases} \tag{5}$$

$\mathbf{B}_n$ is an orthonormal block diagonal matrix with $2(n+1)$ rows of $2(n+1)$ blocks, each of size $M \times M$.

$$\mathbf{B}_n = diag\,\{\,\Theta_n,\,\Theta_n,\,\cdots,\,\Theta_n\,\} \tag{6}$$

$$\Theta_n = \begin{bmatrix} -\mathbf{C}_n & \mathbf{S}_n\mathbf{J}_{M/2} \\ \mathbf{J}_{M/2}\mathbf{S}_n & \mathbf{J}_{M/2}\mathbf{C}_n\mathbf{J}_{M/2} \end{bmatrix} \tag{7}$$

$$\mathbf{C}_n = diag\,\{\cos(\theta_{0,n})\ \cos(\theta_{1,n})\cdots\cos(\theta_{M/2-1,n})\}$$

$$\mathbf{S}_n = diag\,\{\sin(\theta_{0,n})\ \sin(\theta_{1,n})\cdots\sin(\theta_{M/2-1,n})\}$$

$\theta$ are the rotation angles and free parameters in the design of an ELT [3], while $\mathbf{J}$ is the counterdiagonal matrix [3] (ones only in the counter diagonal). For an infinite (at least long enough) input-output sequence (vectors $\mathbf{X}$ and $\mathbf{Y}$ respectively), and using lapped transforms, we have the relation [2]

$$\mathbf{Y} = \tilde{\mathbf{P}}\mathbf{X} \text{ and } \mathbf{X} = \tilde{\mathbf{P}}^T\mathbf{Y} \tag{8}$$

where $\tilde{\mathbf{P}}$ is a banded block circulant matrix. For ELTs, by (1), (6) and (7), we can express $\tilde{\mathbf{P}}$ as

$$\tilde{\mathbf{P}} = \tilde{\mathbf{Z}}\,\tilde{\mathbf{D}}_0\,\tilde{\mathbf{B}}_0\,\tilde{\mathbf{D}}_1\,\tilde{\mathbf{B}}_1\,\cdots\,\tilde{\mathbf{D}}_{K-1}\,\tilde{\mathbf{B}}_{K-1} \tag{9}$$

Fig. 1 shows the flow-graph for a fast ELT with $K = 2$. It follows the non-causal representation typical from the transform matrix viewpoint. In this figure, the order of the input-output $M$-sample blocks is indicated. Each branch in this flow-graph carries $M/2$ samples and input is led to output by straight lines and orthogonal building blocks. Therefore, the inverse can be accomplished by reverting the flow-graph, i.e. following the paths from right to left in Fig. 1. From Fig. 1, one can find $\mathbf{P}$ by isolating the paths connecting eight $M/2$-sample time blocks centered on block $k$, to the output frequency block $k$. With this procedure, it could be possible to reverse the steps in our presentation and given the flow-graph (or the overlapped matrix $\tilde{\mathbf{P}}$) find the corresponding transform matrix $\mathbf{P}$ (or equivalently the filters of the bank). The reader can also check the flow-graph against the definitions for the $\mathbf{P}$ matrix, in the case of an ELT for $K = 2$.

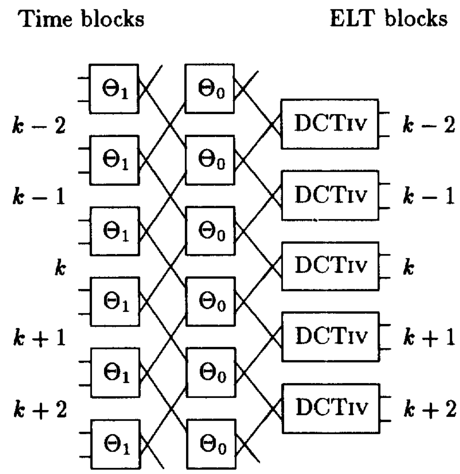


Figure 1. Flow graph for the ELT with $K = 2$. Time and ELT domain samples are grouped into blocks of $M$ samples.

# 3 PR Time Varying ELT

The basic idea for combining variation and PR relies on the following: why can we use the same flow-graph (as example in Fig. 1) for both analysis and synthesis with PR? The answer is because the elements are orthonormal and the reordering paths form a big orthogonal permutation matrix. Since any orthogonal transform is a sucession of plane rotations and have the rotation angles as the only free parameters, what happens if these parameters are changed along time axis as depicted in Fig. 2 for our previous exmaple ? In this case each block would be still orthonormal. Clearly, nothing is changed regarding the relation analysis-synthesis, i.e., the same analysis flow-graph can be used for synthesis yielding PR. Therefore, one can change the angles and switch between two different designs of ELT and keep changing it whenever necessary or desirable. At each instant $k$ we can compute the instantaneous transform matrix $\mathbf{P}(k)$ and the same transform matrix will be used to reconstruct the signal at instant $k$ at the synthesis side (while the fast algorithm is inherently maintained) from:

$$\mathbf{P}(k) = \mathbf{Z}(k)\,\mathbf{D}_0\,\mathbf{B}_0(k)\,\mathbf{D}_1\,\mathbf{B}_1(k)\,\cdots\,\mathbf{D}_{K-1}\,\mathbf{B}_{K-1}(k) \tag{10}$$

Based on this principle and using the fact that any lapped transform (paraunitary filter bank) can be fac-

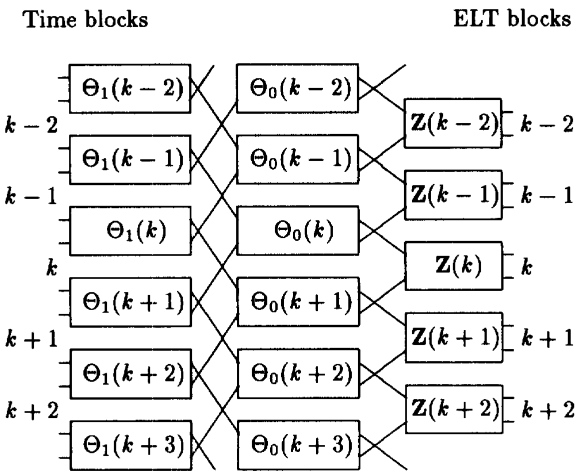


Figure 2. Flow graph for the time-varying ELT with $K = 2$. Time and ELT domain samples are grouped into blocks of $M$ samples.

tored into a cascade of $N$ orthogonal factors composed by plane rotations [7], in [8] it is shown that it is possible to find a structure to implement any lapped transform and commute between them. Since $\mathbf{P}(k)$ is no longer constant with $k$, the PR conditions must ensure the orthogonality among $\mathbf{P}(k)$ and its neighbours $\mathbf{P}(k \pm 1), \mathbf{P}(k \pm 2), \mathbf{P}(k \pm 3) \ldots$ Then, it is easy to show that the PR conditions for time-varying lapped transforms (paraunitary filter banks) are

$$\begin{aligned} \sum_{m=0}^{N-1-\ell} \mathbf{P}_m(k)\mathbf{P}^T_{m+\ell}(k-\ell) &= \\ \sum_{m=0}^{N-1-\ell} \mathbf{P}_{m+\ell}(k)\mathbf{P}^T_{m}(k+\ell) &= \delta(\ell)\mathbf{I}_M \end{aligned} \quad (11)$$

for $\ell = 0, 1, \ldots, N-1$, yielding $2N-1$ independent matrix equations.

As a remark, the term lapped transform was maintained because $\tilde{\mathbf{P}}$ remains orthogonal and for each instant $k$, the synthesis filters are time-reversed versions of the analysis ones. Using the same reasoning, a filter bank can be said *instantaneously paraunitary* if (11) holds for all $k$.

The parameters of an ELT are the rotation angles in the $\Theta_i$ matrices. They also define the low-pass linear-phase prototype that is modulated in order to create the filter bank. As an example for $K = 1$ we have the relations between the window (prototype) and variable angles as

$$\begin{aligned} &h(\ell,k) = -\cos(\theta_{\ell,0}(k)) \\ &h(M-1-\ell,k) = -\sin(\theta_{\ell,0}(k)) \\ &h(M+\ell,k) = -\sin(\theta_{\ell,0}(k+1)) \\ &h(2M-1-\ell,k) = -\cos(\theta_{\ell,0}(k+1)) \\ &\ell = 0,1,\ldots M/2-1 \end{aligned} \quad (12)$$

If $\mathbf{P}(k-1) \neq \mathbf{P}(k) \neq \mathbf{P}(k+1)$, $\mathbf{P}(k)$ is a transitory filter bank. Maybe it would be a good idea if we just switch between two filter banks at a time in order to simplify the design and understanding of the variation process. Whenever there is a switch between two filter banks, there will be a transition region between the steady states as also pointed in [4]. For a general commutation this transitory state would last for $K+1$ blocks, before the steady frequency response of the second filter bank is achieved. The transitory frequency response can be undesirable, but we can try to improve it by finding a new matrix $\mathbf{Z}(k)$ to replace the DCT in (2). For this, from (10), we can write $\mathbf{P}(k) = \mathbf{Z}(k)\mathbf{W}(k)$ and let the input process $x(n)$, for instant $k$, be represented by the vector $\mathbf{x}$ with autocorrelation matrix $\mathbf{R}_{xx}$. If $\mathbf{w} = \mathbf{W}(k)\mathbf{x}$ and $\mathbf{y} = \mathbf{P}(k)\mathbf{x} = \mathbf{Z}(k)\mathbf{w}$, then

$$\mathbf{R}_{ww}(k) = \mathbf{W}(k)\ \mathbf{R}_{xx}\ \mathbf{W}^T(k) \quad (13)$$

$$\mathbf{R}_{yy}(k) = \mathbf{Z}(k)\ \mathbf{R}_{ww}\ \mathbf{Z}^T(k) \quad (14)$$

where $\mathbf{R}_{ww}$ and $\mathbf{R}_{yy}$ are the autocorrelation matrices of the processes associated with $\mathbf{w}$ and $\mathbf{y}$, respectively. $\mathbf{R}_{ww}$ is known a priori (given the estimates of the autocorrelation functions). It is well known that an optimal orthogonal matrix $\mathbf{Z}(k)$ for decorrelating $\mathbf{y}$ would have its rows as the $M$ eigenvectors of $\mathbf{R}_{ww}$. In compression-coding applications, decorrelation is always desired. However, the better the filters, the lower the correlation among subbands. We, therefore, should expect good frequency response using make-up matrices that lead to decorrelation of input signal. This concept was also used in the development of the LOT [3].

# 4 Possible Applications

**Variable Overlapping-** The function that modulates the cosine waveform, in order to generate the filters, can also be viewed as a window weighting all the basis functions, which, otherwise would be sinusoids of different frequencies. Therefore, if we set marginal elements of the window to zero it is possible to actively change the amount of overlap of the transform. For the case

$K = 1$ this is easily achievable by setting some angles to be $\pi/2$. However, a carefull investigation over the overall frequency response of the filter bank, has to be carried, since the window will actually correspond to a filter that will be modulated to create the band-pass filters in the bank and the counterpart of setting elements in the extremes to be zero implies a flat window in the middle. Actually, the window can be made very small in the extremes, and not necessarily zero. In image coding, one can, for example, chose 3 or 4 distinct designs and switch between them for different regions of an image such as: edges, different texture patterns, slow variations, etc... The intention is to minimize artifacts such as blocking and ringing.

**PR Adaptive Wavelet Packets-** A general formulation for orthogonal wavelet packets can come from the hierarchical association of paraunitary filter banks following the paths of an $M$-ary tree. To implement a time-varying wavelet packet we can use a *transparent* state of an ELT. In this state the angles are all chosen as $\pi/2$ and

$$\mathbf{Z} = \begin{pmatrix} \mathbf{J}_{M/2} & \mathbf{0}_{M/2} \\ \mathbf{0}_{M/2} & \mathbf{J}_{M/2} \end{pmatrix}^{K} \begin{pmatrix} \mathbf{0}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & \mathbf{0}_{M/2} \end{pmatrix}^{K+1} \quad (15)$$

This state of an ELT will force the $M$-samples input block to be copied to output unchanged, bypassing the transform. The reason for using this method instead of just copying the input samples to output lies on the fact that it is possible to adaptively activate or bypass a filter bank, while *PR is inherently maintained in the transitions.* If we apply this concept for the wavelet packets case, it would be possible to prune and expand the tree-branches of the hierarchical connection in an adaptive and continuous way. Therefore, it would be possible to construct a PR time-varying wavelet packets structure that will adaptively shape the tree according to measurements in the input signal. More on this subject can be found in [9], where an adaptation algorithm is also discussed. The time-varying partition of the time-frequency plane using a variable shape tree was also studied in [10], where good results were obtained for speech coding. This can be a promising application for this approach.

## 5 Conclusion

We have presented a structure for time-varying filter banks which guarantees distortionless processing. At this step, we just intend to outline basic principles of this method applied to ELTs (general case can be found in [8]). More intensive efforts on design and applications are reserved for further research. The reader shall notice that using this method it is not possible to change the number of channels ($M$) and for $K > 1$ it is very difficult to change the filters' length. However, the change to a bypass state can lead to an adaptive wavelet packets, and, therefore, to an uncountable number of uniform and non-uniform filter banks.

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