SUBBAND PROCESSING OF FINITE LENGTH SIGNALS WITHOUT BORDER DISTORTIONS

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Abstract

New equations are derived for the perfect reconstruction of the boundary regions of a subband processed image. These equations are valid for filter banks with an arbitrary number of filters, having arbitrary non-linear phase, and allowing any border extension method in analysis section. The filter bank is maximally decimated in a wide sense, i.e., the sum of samples among the subbands is equal to the sum of the image samples, without having to store extended subband signals.

1. Introduction

Once one is concerned with the implementation of a maximally decimated FIR filter bank for subband coding of images, the processing of the boundary region of the image can lead us to some interesting problems. Suppose it is desired to use a bank of M FIR filters for processing a finite one-dimensional signal of NM samples. Let $x(n)$ and $\hat{x}(n)$ be the original and recovered signals (after analysis, processing, and synthesis), respectively. Assume also that the filters have length $L = 2KM$ and impulse responses $h_k(n)$ (k=0,1,...,M-1). Since the bank is maximally decimated, each subband must have N samples. Some samples outside the image must be computed in the filtering process and the signal resulting from the convolution of $x(n)$ and $h_k(n)$ would have $NM + L - 1$ samples. Including the decimation process, for each border, we have that $(2K-1)M/2$ samples outside the signal region of support must be computed, and K samples, for each subband, would be deleted. Therefore, $(2K-1)M/2$ samples for each border of the reconstructed signal $\hat{x}(n)$ would be affected by the distortion caused by those missing sub-band samples. Fig. 1 illustrates this process, indicating distortion regions.

In [1] this problem was solved for a DFT implementation, with the transmission of few samples of $x(n)$ as overhead. When a DFT is used the signal is assumed to be periodical, and the discontinuities in the border will produce some high frequency components. In [2], it was studied how $x(n)$ could be extended in order to minimize the distortion in $\hat{x}(n)$. Typical extensions are obtained by symmetrical reflection and repetition of the border sample. However, it is quite easy to see that if the reflection is symmetric and also the filters, the deleted subband samples could be recovered by a simple symmetric reflection of this signal, and $\hat{x}(n)$ will suffer no distortion. This fact was recognized in [3]. All these approaches used $M = 2$ and linear phase filters. We will present results for M-band non-linear phase systems.

2. Notation and Definitions

For simplicity, we will discuss the one-dimensional case only, assuming that an image would be processed in a row-column implementation. Assuming that a row is to be processed, we will focus our attention on the left border. Let $h_k(L-1-n)$, $n = 0, 1 \ldots, L-1$, $k = 0, 1 \ldots, M-1$, be the rows of the $M \times L$ transform matrix $\mathbf{H}$. Divide $\mathbf{H}$ into $M \times M/2$ matrices

$$\mathbf{H} = [\mathbf{H}_0\ \mathbf{H}_1\ \mathbf{H}_2\ \cdots \mathbf{H}_{4K-1}] \quad (1)$$

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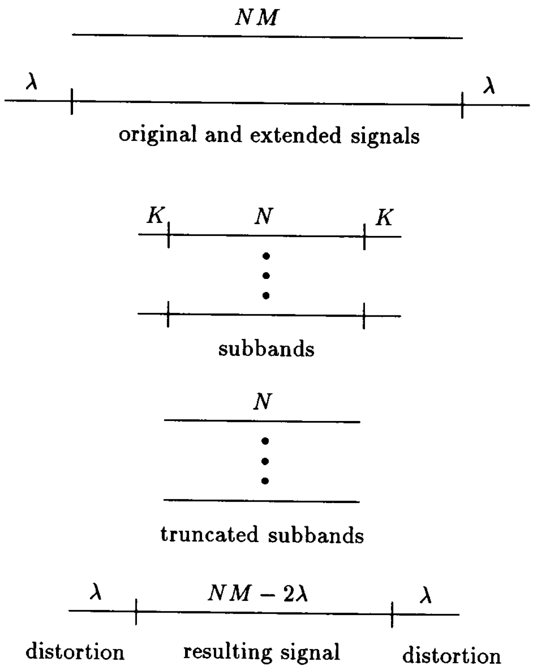

Figure 1: Illustration of border distortions caused by truncation of subband signals. In this figure filters are assumed to have length $L = 2KM$ and $\lambda = (L-M)/2$.

Divide the input and recovered signals $x(n)$ and $\hat{x}(n)$ into $M/2$-sample blocks and suppose them to have infinite length,

$$\mathbf{X}^T = \cdots\ \mathbf{X}_{-3}^T\,\mathbf{X}_{-2}^T\,\mathbf{X}_{-1}^T\ |\ \mathbf{X}_0^T\,\mathbf{X}_1^T\,\mathbf{X}_2^T\,\mathbf{X}_3^T \cdots \qquad (2)$$

$$\hat{\mathbf{X}}^T = \cdots\ \hat{\mathbf{X}}_{-3}^T\,\hat{\mathbf{X}}_{-2}^T\,\hat{\mathbf{X}}_{-1}^T\ |\ \hat{\mathbf{X}}_0^T\,\hat{\mathbf{X}}_1^T\,\hat{\mathbf{X}}_2^T\,\hat{\mathbf{X}}_3^T \cdots \qquad (3)$$

In these equations the traces indicate the borders and superscript T denotes matrix transposition. The blocks $\mathbf{X}_i$, $i < 0$, should be found from $\mathbf{X}_i$, $i \geq 0$, by an extension method. The transformed infinite vector $\mathbf{Y}$ containing the subband samples can be also partitioned into M-sample blocks, containing one sample from each subband,

$$\mathbf{Y}^T = \cdots\ \mathbf{Y}_{-3}^T\,\mathbf{Y}_{-2}^T\,\mathbf{Y}_{-1}^T\ |\ \mathbf{Y}_0^T\,\mathbf{Y}_1^T\,\mathbf{Y}_2^T\,\mathbf{Y}_3^T \cdots \qquad (4)$$

$$\hat{\mathbf{Y}}^T = \cdots\ \hat{\mathbf{Y}}_{-3}^T\,\hat{\mathbf{Y}}_{-2}^T\,\hat{\mathbf{Y}}_{-1}^T\ |\ \hat{\mathbf{Y}}_0^T\,\hat{\mathbf{Y}}_1^T\,\hat{\mathbf{Y}}_2^T\,\hat{\mathbf{Y}}_3^T \cdots \qquad (5)$$

Hence,

$$\mathbf{Y}_n = \sum_{i=0}^{4K-1} \mathbf{H}_i \mathbf{X}_{i-(2K-1)+2n} \qquad (6)$$

If we desire a somewhat smooth transition across the borders, we could find the blocks out of the row as a linear function of the blocks inside it. Remember that $2K-1$ blocks of $\mathbf{X}$ outside the row are necessary for computing $\mathbf{Y}_i$, $i \geq 0$. Let $\mathbf{X}_R^T = [\mathbf{X}_{-2K+1}^T\ \ldots\ \mathbf{X}_{-1}^T]$ and $\mathbf{X}_D^T = [\mathbf{X}_0^T\ \ldots\ \mathbf{X}_{2K-2}^T]$. We will assume that

$$\mathbf{X}_R = \mathbf{R}\mathbf{X}_D \qquad (7)$$

Denoting $\mathbf{J}_k$ as the counter-identity matrix of order k,

$$\mathbf{J}_k = \begin{pmatrix} 000\cdots01 \\ 000\cdots10 \\ \vdots\quad\ \vdots \\ 010\cdots00 \\ 100\cdots00 \end{pmatrix}$$

We have that, for a symmetric reflection around the borders, we must chose

$$\mathbf{R} = \mathbf{J}_{(2K-1)M/2} = \mathbf{J}_{(L-M)/2} \qquad (8)$$

This kind of reflection would guarantee polyphase normality (i.e., a flat input signal generates only one non-zero frequency component) and avoid discontinuities across the borders. All results presented here may be extended to the right border, replacing $\mathbf{X}, \mathbf{Y}$ and $\mathbf{H}$ by $\bar{\mathbf{X}}, \bar{\mathbf{Y}}$ and $\bar{\mathbf{H}}$, respectively, with

$$\begin{aligned} \bar{\mathbf{X}}_i &= \mathbf{X}_{2N-1-i}\mathbf{J}_{M/2} \\ \bar{\mathbf{Y}}_i &= \mathbf{Y}_{N-1-i}\mathbf{J}_{M/2} \\ \bar{\mathbf{H}}_i &= \mathbf{H}_{4K-1-i}\mathbf{J}_{M/2} \quad i = 0, 1 \ldots, 4K-1 \end{aligned}$$

3. Linear Phase Filters

If the filters in the filter bank are symmetric or anti-symmetric, then $h_k(L-1-n) = v_k h_k(n)$, where $v_k = \pm 1$ depending on the symmetry. Let $\mathbf{V} = \mathrm{diag}\{v_0, v_1, \ldots, v_{M-1}\}$ and note that $\mathbf{V}^{-1} = \mathbf{V}$. It is easy to verify that in (1), for $i = 0, \ldots, 4K-1$, we have

$$\mathbf{H}_{4K-1-i} = \mathbf{V}\mathbf{H}_i\mathbf{J}_{M/2} \tag{9}$$

For instance, assume that $x(n)$ is infinite and that samples are folded around the border. Since $\mathbf{JJ} = \mathbf{I}$, and using the reflection in (8), it follows that $\mathbf{X}_{-n-1} = \mathbf{J}_{M/2}\mathbf{X}_n$, for any integer n. Thus, for any n, from (6),

$$\begin{aligned}\mathbf{Y}_{-n-1} &= \sum_{i=0}^{4K-1} \mathbf{H}_i\mathbf{X}_{i-(2K-1)-2n-2} \\ &= \sum_{i=0}^{4K-1} \mathbf{H}_i\mathbf{J}\mathbf{X}_{-i+2K+2n}\end{aligned}$$

If we revert the order of summation and apply (9), we get

$$\mathbf{Y}_{-n-1} = \sum_{i=0}^{4K-1} \mathbf{V}\mathbf{H}_i\mathbf{X}_{i-(2K-1)+2n} = \mathbf{V}\mathbf{Y}_n \tag{10}$$

This means that, for any integer n, $\mathbf{Y}_{-n-1} = \mathbf{V}\mathbf{Y}_n$ if $\mathbf{X}_{-n-1} = \mathbf{J}_{M/2}\mathbf{X}_n$. Therefore, for positive i and k, the algorithm for perfect reconstruction of the borders is straightforward: (i) Find $\mathbf{X}_{-i-1}$ from $\mathbf{X}_i$; (ii) Calculate $\mathbf{Y}$; (iii) Process positive indices of $\mathbf{Y}$; (iv) From $\hat{\mathbf{Y}}_k$, find $\hat{\mathbf{Y}}_{-k-1}$ using (10); (v) Calculate $\hat{\mathbf{X}}$ from $\hat{\mathbf{Y}}$. Actually, $i = 0, \ldots, 2K-1$ and $k = 0, \ldots, K-1$. For the special case of two-band QMF banks, $v_0 = 1$, $v_1 = -1$ and $\mathbf{J}=[1]$. This solution is insensitive to quantization errors, in a sense that distortions in the borders caused by coarse quantization of $\mathbf{Y}$ have the same magnitude and nature as the distortions in other regions of the signal.

4. Non-Linear Phase Filters

Divide the signal $\mathbf{X}$ into three vectors. The reflected blocks, $\mathbf{X}_R$, the blocks that would suffer distortion in the synthesis procedure, $\mathbf{X}_D$, and the blocks that would be perfectly reconstructed, $\mathbf{X}_P^T = [\mathbf{X}_{2K-1}^T\mathbf{X}_{2K}^T\cdots]$. The first two were previously defined and obey (7). Equation (6) is still valid, and for $n = 0, 1, \ldots, 2K-2$ one can divide the sum into three components, one for each vector,

$$\begin{aligned}\mathbf{Y}_n = &\sum_{i=0}^{2K-2-2n} \mathbf{H}_i\mathbf{X}_{i-2K+1+2n} \\ &+ \sum_{i=2K-1-2n}^{4K-3-2n} \mathbf{H}_i\mathbf{X}_{i-2K+1+2n} \\ &+ \sum_{i=4K-2-2n}^{4K-1} \mathbf{H}_i\mathbf{X}_{i-2K+1+2n}\end{aligned} \tag{11}$$

The above equation can be put in matrix notation as

$$\mathbf{Y}_n = \mathbf{F}_n^{(1)}\mathbf{R}\mathbf{X}_D + \mathbf{F}_n^{(2)}\mathbf{X}_D + \mathbf{F}_n^{(3)}\mathbf{X}_P \tag{12}$$

where

$$\mathbf{F}_n^{(1)} = [\mathbf{0}\ \mathbf{0}\cdots\mathbf{0}\ \mathbf{H}_0\cdots\mathbf{H}_{2K-2-2n}]$$

$$\mathbf{F}_n^{(2)} = [\mathbf{0}\ \mathbf{0}\cdots\mathbf{0}\ \mathbf{H}_{2K-1-2n}\cdots\mathbf{H}_{4K-3-2n}]$$

$$\mathbf{F}_n^{(3)} = [\mathbf{H}_{4K-2-2n}\cdots\mathbf{H}_{4K-1}\ \mathbf{0}\ \mathbf{0}\cdots\mathbf{0}]$$

The zero matrices are padded into the vectors to align them in the above equation, as in the example in Fig. 2 for $K = 2$. Grouping $\mathbf{F}_n^{(i)}$ and $\mathbf{Y}_n$ for $n = 0, 1, \ldots, 2K-2$, we define the matrices Φ_i (see Fig. 2 for $K = 2$) and the vector Υ as

$$\Upsilon = \begin{pmatrix}\mathbf{Y}_0 \\ \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_{2K-2}\end{pmatrix} \qquad \Phi_i = \begin{pmatrix}\mathbf{F}_0^{(i)} \\ \mathbf{F}_1^{(i)} \\ \vdots \\ \mathbf{F}_{2K-2}^{(i)}\end{pmatrix}$$

Hence, (12) can be rewritten as

$$\Upsilon = (\Phi_1\mathbf{R} + \Phi_2)\mathbf{X}_D + \Phi_3\mathbf{X}_P \tag{13}$$

Let $\mathbf{A} = \Phi_1\mathbf{R} + \Phi_2$ and $\mathbf{B} = \Upsilon - \Phi_3\mathbf{X}_P$. Thus (13) reduces to

$$\mathbf{A}\mathbf{X}_D = \mathbf{B} \tag{14}$$

The matrix $\mathbf{A}$ has dimensions $(4K-2)M/2 \times (2K-1)M/2$. If $\mathbf{A}$ has maximum rank $(2K-1)M/2$, we can left-multiply it by a matrix $\mathbf{S}$ in order to make $\mathbf{SA}$ a non-singular invertible square matrix of order $(2K-1)M/2$. In this case the solution is given by

$$\mathbf{X}_D = (\mathbf{SA})^{-1}\mathbf{SB} \tag{15}$$

$$[\Phi_1|\Phi_2|\Phi_3] = \left(\begin{array}{ccc|ccc|cccccc} \mathbf{H}_0 & \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 & \mathbf{H}_4 & \mathbf{H}_5 & \mathbf{H}_6 & \mathbf{H}_7 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_0 & \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 & \mathbf{H}_4 & \mathbf{H}_5 & \mathbf{H}_6 & \mathbf{H}_7 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_0 & \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 & \mathbf{H}_4 & \mathbf{H}_5 & \mathbf{H}_6 & \mathbf{H}_7 \end{array}\right)$$

Figure 2: Example of Φ_i matrix grouping procedure for $K = 2$.

If one choose $\mathbf{S}$ as the pseudo-inverse of $\mathbf{A}$, $\mathbf{S} = \mathbf{A}^P = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$, and if $\mathbf{A}^P\Phi_3 = \mathbf{0}$, then

$$\mathbf{X}_D = \mathbf{A}^P\Upsilon \tag{16}$$

Regrettably, we were not able to determine the rank of $\mathbf{A}$ in a explicit way. However, we have made tests with several perfect reconstruction cosine modulated filter banks [4], for several values of M and K. All of them lead to $\mathbf{A}$ with maximum rank $(L - M)/2$, and to $\mathbf{A}^P\Phi_3 = \mathbf{0}$.

The correct choice of $\mathbf{S}$ is essential to coding systems. It can be any matrix leading to a non-singular matrix $\mathbf{SA}$, but different choices of $\mathbf{S}$ would lead to solutions with distinct sensitivity to quantization errors. For a sufficiently large bit-rate, all valid $\mathbf{S}$ will result in no visible distortion. In applications requiring no compression, the method is inherently distortion-free. A good choice for $\mathbf{S}$ is $\mathbf{S} = \mathbf{A}^P$, which generally results in lower sensitivity to quantization errors than $\mathbf{S}$ matrices that select sets of $(2K-1)M/2$ independent rows of $\mathbf{A}$.

Taking a simple example, the case of K=1 and M=2 (a 4-tap perfect reconstruction two-channel filter bank), we have $\mathbf{H}_n^T = [h(n)\ h(3-n)(-1)^n]$ [5]. The $\mathbf{X}$ blocks reduce to scalars and $\mathbf{Y}_0^T = [y_L\ y_H]$. Assuming $\mathbf{X}_{-1} = \mathbf{X}_0 = x(0)$, and $h(n) = [-0.14\ 0.35\ 0.85\ 0.35]$ (approximately), we find that (13) and (16) reduce to

$$x(0) = \begin{cases} 4.83y_L - 4.12x(1) - 1.71x(2) \\ -2.00y_H + 0.71x(1) + 0.29x(2) \\ 0.707y_L - 1.707y_H \end{cases}$$

These equations are solutions for $\mathbf{S}$ as [1 0], [0 1] and $\mathbf{A}^P$, respectively. It can be seen that the first equation is less robust, since the random quantization errors on $\hat{y}$ and $\hat{x}$ would be amplified. In an image coding test, the last solution led to less visible artifacts.

5. Conclusion

We have derived equations for the perfect reconstruction of the boundary regions of a finite sample vector, processed by a maximally decimated filter bank. These equations are general and independent of the signal. Further studies are necessary in order to determine the matrices $\mathbf{S}$ and $\mathbf{R}$ that minimize the quantization error effects over $\mathbf{X}_D$. The reflection matrix can also be chosen arbitrarily, but it is preferable to not introduce high frequencies components in the analysis process, allowing discontinuities in the extended signal.

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